ALMOST ISOMETRIC MAPS OF THE HYPERBOLIC PLANE

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1. *Introduction*

The hyperbolic distance between points p and q in the open unit disc D is

$$d(p,q) = \inf_{\gamma} \int_{\gamma} \frac{2|dz|}{1-|z|^2},$$

where the infimum is over all arcs γ in D joining p to q. If $\mathcal{M}$ denotes the group of conformal self maps

$$Tz = \lambda \frac{z-a}{1-\bar{a}z}, \qquad a \in D, |\lambda| = 1,$$

of D, then

$$d(Tp, Tq) = d(p,q)$$

for all $T \in \mathcal{M}$; thus maps in $\mathcal{M}$ are hyperbolic isometrics. The Schwarz–Pick theorem asserts that if $f: D \to D$ is analytic then f decreases distances,

$$d(f(p), f(q)) \leqslant d(p,q), \tag{1.1}$$

or infinitesimally,

$$\frac{|f'(p)|(1-|p|^2)}{1-|f(p)|^2} \leqslant 1. \tag{1.2}$$

Equality anywhere in (1.1) or (1.2) implies that $f \in \mathcal{M}$ and then equality holds everywhere.

Fix a constant $c > 0$. Following C. McMullen, we write $M(c)$ for the set of analytic $f: D \to D$ such that whenever B is a hyperbolic ball in D,

$$\operatorname{diam}(f(B)) \geqslant \operatorname{diam}(B) - c,$$

where diam denotes diameter in the hyperbolic metric. For example,

$$\bigcap_{c>0} M(c) = \mathcal{M},$$

while $f(z) = z^N \in M(c)$ provided c is large. This paper gives three characterizations of the set $M(c)$. The first characterization concerns nearly isometric behavior along certain geodesics, and the second is in terms of angular derivatives at boundary points. Each $f \in M(c)$ is a Blaschke product, and the third characterization is by the distribution of the zeros. We thank Curt McMullen for bringing $M(c)$ to our attention and for the results of the next section.

2. *First properties of* $M(c)$

By the invariance of the hyperbolic metric we clearly have

$$f \in M(c) \quad \text{if and only if} \quad T \circ f \circ S \in M(c) \quad \text{for all} \quad T, S \in \mathcal{M}. \tag{2.1}$$

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Suppose that $f \in M(c)$. Then by Fatou's theorem f has angular limit $f(\zeta)$ at almost all $\zeta \in \partial D$. Condition $M(c)$ implies that $|f(\zeta)| = 1$.

LEMMA 2.1. *Suppose that $f \in M(c)$ and suppose that σ is an arc in D with end point $\zeta \in \partial D$. If*

$$\lim_{\sigma \ni z \to \zeta} f(z) = \alpha$$

exists, then $|\alpha| = 1$.

Proof. Since f is bounded, Lindelöf's theorem gives

$$\lim_{\Gamma \ni z \to \zeta} f(z) = \alpha$$

for every cone $\Gamma = \Gamma(K) = \{z : |z-\zeta| < K(1-|z|\}, K > 0$. Fix $R > \frac{1}{2}c$ and for $0 < r < 1$ set $B_r = \{z : d(z, r\zeta) < R\}$. Then there is $K = K(R)$ such that $B_r \subset \Gamma(K)$ for $1-r$ small and such that

$$\lim_{r \to 1} \sup_{B_r} |z-\zeta| = 0.$$

Hence

$$\lim_{r \to 1} \sup_{B_r} |f(z)-\alpha| = 0.$$

If $|\alpha| < 1$, then $\lim \operatorname{diam}(f(B_r)) = 0$ while $\operatorname{diam} B_r = 2R > c$, a contradiction to $M(c)$.

By convention we call $f(z) = \lambda B(z)$ a *Blaschke product* if $B(z)$ is a Blaschke product and $|\lambda| = 1$.

COROLLARY 2.2. *If $f \in M(c)$, then $T \circ f \circ S$ is a Blaschke product for all $T, S \in \mathcal{M}$.*

Proof. By (2.1) it is enough to prove that f is a Blaschke product. By Lemma 2.1 f is an inner function: $|f(\zeta)| = 1$ almost everywhere on ∂D. Every inner function is a Blaschke product times a singular function and every singular function has radial limit 0 at some $\zeta \in \partial D$, see [**4**, p. 76]. So if the singular factor were non-constant, f would also have radial limit 0 at ζ, contradicting the lemma.

A theorem of Frostman says that every inner function has the form $T \circ f$ with $T \in \mathcal{M}$ and f a Blaschke product. So there are many Blaschke products not in any $M(c)$.

3. *Geodesic condition*

The geodesics in the hyperbolic metric are the arcs of circles and lines orthogonal to ∂D. Write (p, q) for the unique geodesic are joining the points $p, q \in \bar{D}$.

THEOREM 3.1. *There exist $\rho = \rho(c)$ and $\delta = \delta(c)$ such that if $f \in M(c)$, then for all $z \in D$ there is a geodesic σ such that*

$$\operatorname{dist}(z, \sigma) = \inf\{d(z, p) : p \in \sigma\} \leqslant \rho \tag{3.1}$$

and

$$d(f(p), f(q)) \geqslant d(p, q) - \delta \tag{3.2}$$

for all $p, q \in \sigma$. Conversely, if $\rho > 0$ and $\delta > 0$ there is $c = c(\rho, \delta)$ such that $f \in M(c)$ if for every $z \in D$ there is a geodesic σ satisfying (3.1) *and* (3.2).

Proof. Assume that $f \in M(c)$. Since (3.1) and (3.2) are conformally invariant, we may assume that $z = 0$ and $f(0) = 0$. Then there are z_n and w_n such that

$$d(z_n, 0) = n, \qquad d(w_n, 0) = n, \qquad d(f(z_n), f(w_n)) \geqslant 2n - c.$$

By the Schwarz–Pick theorem,

$$d(z_n, w_n) \geqslant 2n - c$$

and the angle $\theta_n \leqslant \pi$ between $(0, z_n)$ and $(0, w_n)$ satisfies

$$\cos\theta_n = \frac{\cosh^2(n) - \cosh d(z_n, w_n)}{\sinh^2(n)}$$

by [1, p. 148]. Hence

$$\cos\theta_n \leqslant 1 - 2e^{-c} + O(e^{-n})$$

and there is $\theta(c) > 0$ such that

$$\underline{\lim}\, \theta_n \geqslant \theta(c).$$

Take subsequences so that $z_n \to \zeta \in \partial D, w_n \to \partial D$. Then $|\zeta - \omega| \geqslant 2\sin(\frac{1}{2}\theta(c))$, and the geodesic $\sigma = (\zeta, \omega)$ satisfies (3.1) with ρ determined by $\theta(c)$.

To prove (3.2), let $p, q \in \sigma$. There are p_n and q_n in (z_n, w_n) such that $p_n \to p$ and $q_n \to q$. Say p_n falls between z_n and q_n on (z_n, w_n). Then

$$\begin{aligned} d(f(p_n), f(q_n)) &\geqslant d(f(z_n), f(w_n)) - d(z_n, p_n) - d(w_n, q_n) \\ &\geqslant d(z_n, w_n) - c - d(z_n, p_n) - d(q_n, w_n) \\ &= d(p_n, q_n) - c. \end{aligned}$$

Thus (3.2) holds with $\delta = c$.

Conversely, let $R > \rho$ and set $B = \{w : d(w, z) < R\}$. When σ satisfies (3.1) and (3.2), $\sigma \cap \partial B = \{p, q\}$ and

$$d(p, q) \geqslant 2R - 2\rho.$$

Then by (3.2)

$$d(f(p), f(q)) \geqslant 2R - 2\rho - \delta.$$

Therefore

$$\operatorname{diam} f(B) \geqslant \operatorname{diam} B - (2\rho + \delta) \tag{3.3}$$

whenever $\operatorname{diam} B > 2\rho$. Since (3.3) is trivial if $\operatorname{diam} B \leqslant 2\rho$ we conclude that $f \in M(c)$ with $c = 2\rho + \delta$.

REMARK. The above proof works because the hyperbolic metric has constant negative curvature. The negative curvature shows up in the inequality $\underline{\lim}\, \theta_n > 0$.

Condition (3.2) is very strong. It implies that f has an angular derivative and a unimodular conical limit of each end point of σ. Moreover, when restricted to a cone at either end point of σ, f is asymptotic to a Möbius transformation.

THEOREM 3.2. *Let σ be the geodesic arc joining $p \in D$ to $\zeta \in \partial D$, let $\delta > 0$, and let f be an analytic map from D to D satisfying*

$$d(f(z), f(w)) \geqslant d(z, w) - \delta \quad \textit{for all } z, w \in \sigma. \tag{3.4}$$

Then there exist $\lambda \in \partial D$ *and* $A, 0 < A \leqslant e^{\delta}$, *such that for every cone*

$$\Gamma = \{z : |z-\zeta| < K(1-|z|)\}, \qquad K > 0,$$

$$\lim_{\Gamma \ni z \to \zeta} f(z) = \lambda \tag{3.5}$$

and

$$\lim_{\Gamma \ni z \to \zeta} \frac{\lambda - f(z)}{\zeta - z} = \lim_{\Gamma \ni z \to \zeta} f'(z) = A\lambda\bar{\zeta}. \tag{3.6}$$

If $g \in \mathcal{M}$ *satisfies* $g(\zeta) = \lambda$ *and* $g'(\zeta) = A\lambda\bar{\zeta}$, *then*

$$\lim_{\Gamma \ni z \to \zeta} d(f(z), g(z)) = 0.$$

When (3.6) holds we say f has *angular derivative* $A\lambda\bar{\zeta}$ at ζ and we write $f'(\zeta) = A\lambda\bar{\zeta}$. By the theorem on the angular derivative (see [4, p. 43]) if

$$\varliminf_{z \to \zeta} \frac{1-|f(z)|}{1-|z|} = A < \infty,$$

then (3.6) and (3.5) hold for some λ and for the same A. It then follows that

$$\sup_{\Gamma(K)} \frac{1-|f(z)|}{1-|z|} \leqslant 2AK \tag{3.7}$$

for every cone $\Gamma(K)$ with vertex of ζ.

Proof. We can suppose that $p = 0, f(p) = 0$ and $\zeta = 1$. For $0 < x < 1$, (3.4) gives

$$d(f(x), 0) = \log \frac{1+|f(x)|}{1-|f(x)|} \geqslant \log \frac{1+x}{1-x} - \delta,$$

so that

$$\varliminf_{x \to 1} \frac{1-|f(x)|}{1-x} \leqslant e^{\delta},$$

and the angular derivative theorem yields (3.5) and (3.6) for some λ and for $A \leqslant e^{\delta}$.

We can suppose that $\lambda = 1$. If $g \in \mathcal{M}$, if $g(1) = 1$ and if $g'(1) = A$, then by (3.6)

$$\lim_{\Gamma \ni z \to 1} \frac{|f(z) - g(z)|}{|1-z|} = 0.$$

Now

$$\tanh\left(\frac{d(f(z), g(z))}{2}\right) = \left|\frac{f(z) - g(z)}{1 - \overline{g(z)} f(z)}\right|$$

$$= \frac{|f(z) - g(z)|}{|1-z|} \left\{\left|\frac{1-\overline{g(z)}}{1-\bar{z}} + \overline{g(z)} \frac{1-f(z)}{1-z} \frac{1-z}{1-\bar{z}}\right|\right\}^{-1},$$

and the expression in braces is bounded away from zero when $z \in \Gamma$ and $|1-z|$ is small. Therefore

$$\lim_{\Gamma \ni z \to 1} d(f(z), g(z)) = 0.$$

4. *Angular derivative condition*

Let I be an arc on ∂D with measure $|I| < \pi$. Let c_I be the center of I and write $z_I = (1 - |I|/2\pi) c_I$. Let f denote an analytic map from D to D.

THEOREM 4.1. *If $f \in M(c)$ then f has angular derivative on a dense subset of ∂D and there is $A = A(c)$ such that, for every arc I with $|I| < \pi$,*

$$\inf_{\zeta \in I} |f'(\zeta)| \leqslant A \frac{(1-|f(z_I)|)}{1-|z_I|}. \tag{4.1}$$

Conversely, there is $c = c(A)$ such that, if (4.1) *holds for every arc I with $|I| < \pi$, then $f \in M(c)$.*

Note that the inequality which is the reverse of (4.1), with a different value A, holds whenever f has angular derivative at $\zeta \in I$. That follows from (3.7).

Before proving Theorem 4.1 we give some lemmas on the hyperbolic derivative

$$\frac{|f'(z)|(1-|z|^2)}{1-|f(z)|^2},$$

which is invariant under Möbius transformations of z or of $f(z)$.

LEMMA 4.2. *Given $R > 0$ and $\varepsilon > 0$ there is $\eta > 0$ such that if*

$$\frac{|f'(z_0)|(1-|z_0|^2)}{1-|f(z_0)|^2} > 1-\eta$$

at $z_0 \in D$, then on $B(z_0, R) = \{w : d(z_0, w) < R\}$,

$$\frac{|f'(w)|(1-|w|^2)}{1-|f(w)|^2} \geqslant 1-\varepsilon \tag{4.2}$$

and

$$|f(w)-F(w)|+|f'(w)-F'(w)| < \varepsilon, \tag{4.3}$$

where $F \in \mathscr{M}$ satisfies $F(z_0) = f(z_0)$ and $\arg F'(z_0) = \arg f'(z_0)$.

Proof. Clearly (4.3) implies (4.2), and a normal family argument yields (4.3).

LEMMA 4.3. *If $z_0 \in D$, if $\zeta \in \partial D$ and if*

$$d(f(p), f(q)) \geqslant d(p,q) - c \tag{4.4}$$

for all $p, q \in (z_0, \zeta)$, then

$$E_\varepsilon = \left\{ z \in (z_0, \zeta) : \frac{|f'(z)|(1-|z|^2)}{1-|f(z)|^2} < 1-\varepsilon \right\}$$

satisfies

$$\int_{E_\varepsilon} \frac{2|dz|}{1-|z|^2} < c/\varepsilon. \tag{4.5}$$

If also $(z_0, \zeta) = (0, 1)$ and $f(0) = 0, f(1) = 1$, then

$$F_\varepsilon = \left\{ x \in (0,1) : \frac{\operatorname{Re} f'(x)(1-x^2)}{1-|f(x)|^2} < 1-\varepsilon \right\}$$

has

$$\int_{F_\varepsilon} \frac{2dx}{1-x^2} < c/\varepsilon. \tag{4.6}$$

Proof. We prove (4.6), which implies (4.5). We have by Theorem 3.2,

$$\lim_{x\uparrow 1} d(x,0)-d(f(x),0)) = \lim_{x\uparrow 1}\log\left(\frac{1+x}{1-x}\frac{1-|f(x)|}{1+|f(x)|}\right) = \log f'(1) \leqslant c,$$

and also

$$\lim_{x\uparrow 1} d(x,0)-d(\operatorname{Re} f(x),0) = \lim_{x\uparrow 1}\log\left(\frac{1+x}{1-x}\frac{1-\operatorname{Re} f(x)}{1+\operatorname{Re} f(x)}\right) = \log f'(1) \leqslant c,$$

because by (3.6)

$$\lim_{x\to 1}\frac{\operatorname{Im} f(x)}{1-x} = 0.$$

Therefore

$$\int_0^1 \left\{1-\frac{\operatorname{Re} f'(x)(1-x^2)}{1-(\operatorname{Re} f(x))^2}\right\}\frac{2dx}{1-x^2} \leqslant c$$

and since the integrand is positive, Chebychev's inequality gives (4.6).

LEMMA 4.4. *Let $\varepsilon > 0$. There is $\delta = \delta(c,\varepsilon)$ such that if* (4.4) *holds for all $p, q \in (z_0, \zeta)$, $z_0 \in D$, $\zeta \in \partial D$ and if*

$$\frac{|f'(z_0)|(1-|z_0|^2)}{1-|f(z_0)|^2} \geqslant 1-\delta,$$

then

$$\left|\arg f'(z_0)-\arg\left(\frac{f(\zeta)-f(z_0)}{1-\bar{f}(z_0)f(\zeta)}\frac{1-\bar{z}_0\zeta}{\zeta-z_0}\right)\right| < \varepsilon.$$

Proof. Set

$$g = \mu\frac{f\left(\dfrac{\lambda z+z_0}{1+\bar{z}_0\lambda z}\right)-f(z_0)}{1-\bar{f}(z_0)f\left(\dfrac{\lambda z+z_0}{1+\bar{z}_0\lambda z}\right)},$$

where

$$\lambda = \frac{(\zeta-z_0)}{1-\bar{z}_0\zeta}, \qquad \mu = \frac{1-\bar{f}(z_0)f(\zeta)}{f(\zeta)-f(z_0)}.$$

Then $g(0) = 0$, $g(1) = 1$ and g satisfies (4.4) in $(0,1)$. By Lemma 4.3, $|\arg g'(x)| < \frac{1}{2}\varepsilon$ for some $x \in (0,1)$ with $d(x,0) \leqslant 2c/\varepsilon = R$. By Lemma 4.2, $|\arg g'(w)-\arg g'(0)| < \frac{1}{2}\varepsilon$ for all $w \in B(0,R)$ if δ is small enough. Hence $|\arg g'(0)| < \varepsilon$. But

$$g'(0) = \lambda\mu f'(z_0)\frac{1-|z_0|^2}{1-|f(z_0)|^2}.$$

LEMMA 4.5. *If $w_0 \in D$ and $\zeta \in \partial D$ and if*

$$d(f(p),f(q)) \geqslant d(p,q)-c$$

for all $(p,q) \in (w_0,\zeta)$ and if $d(z_0 w_0) = d$, then

$$d(f(p),f(q)) \geqslant d(p,q)-(c+4d)$$

for all $(p,q) \in (z_0,\zeta)$.

Proof. For $p \in (z_0, \zeta)$, let p^* be its nearest point in (w_0, ζ). Since the geodesics (z_0, ζ) and (w_0, ζ) are asymptotic,

$$d(p, p^*) \leqslant d(z_0, z_0^*) \leqslant d.$$

Then

$$\begin{aligned} d(f(p), f(q)) &\geqslant d(f(p^*), f(q^*)) - d(p, p^*) - d(q, q^*) \\ &\geqslant d(p^*, q^*) - c - 2d \\ &\geqslant d(p, q) - c - 4d \end{aligned}$$

for all $(p, q) \in (z_0, \zeta)$.

Proof of Theorem 4.1. Assume that $f \in M(c)$ and fix an arc I of ∂D with $|I| < \pi$. By Theorem 3.1 there is z_0 such that $d(z_I, z_0) \leqslant \rho_1(c)$, $1 - |z_0| < (1 - |z_I|)/10$ and $z_0/|z_0| \in I$, and there is a geodesic σ containing z_0 such that (3.2) holds on σ. At least one end point of σ falls in I.

Applying Theorem 3.2 to $T \circ f \circ S$, where $T \in \mathcal{M}$, $T(f(z_0)) = 0$ and $S \in \mathcal{M}$, $S(0) = z_0$, we see that when $z \in (z_0, \alpha)$,

$$\frac{1 - |f(z)|^2}{1 - |z|^2} \leqslant e^c \frac{|1 - \bar{f}(z_0) f(z)|^2}{1 - |f(z_0)|^2} \frac{1 - |z_0|^2}{|1 - \bar{z}_0 z|^2}.$$

When $z \in (z_0, \alpha)$ we also have $|1 - \bar{z}_0 z| \geqslant c_0(1 - |z_0|)$. Therefore

$$|f'(\alpha)| \leqslant c_1 e^c \frac{|f(\alpha) - f(z_0)|^2}{(1 - |f(z_0)|^2)(1 - |z_0|^2)}.$$

Since $d(z_0, z_I) \leqslant \rho_1$,

$$\frac{1 - |f(z_0)|^2}{1 - |z_0|^2} \leqslant c_2 \frac{1 - |f(z_I)|^2}{1 - |z_I|^2},$$

and we shall get (4.1) with $A = c_3 e^c$ provided

$$|f(\alpha) - f(z_0)| \leqslant c_4(1 - |f(z_0)|).$$

But now assume that

$$|f(\alpha) - f(z_0)| > c_4(1 - |f(z_0)|) \tag{4.7}$$

for some large constant c_4. We may also assume that

$$\frac{|f'(z_0)|(1 - |z_0|^2)}{1 - |f(z_0)|^2} \geqslant 1 - \delta, \tag{4.8}$$

with δ very small, because by Lemma 4.3 there are points satisfying (4.8) and lying a bounded hyperbolic distance from z_0. Hence by Lemma 4.4,

$$\left| \arg f'(z_0) - \arg\left(\frac{f(\alpha) - f(z_0)}{1 - \bar{f}(z_0) f(\alpha)}\right) + \arg\left(\frac{\alpha - z_0}{1 - \bar{z}_0 \alpha}\right) \right| < \varepsilon. \tag{4.9}$$

Replacing I by $J \subset I$, $|J| = \text{const.}\, |I|$, we can find another geodesic arc (w_0, β) such that $\beta \in J$ and

$$d(f(p), f(q)) \geqslant d(p, q) - c$$

for $p, q \in (w_0, \beta)$, such that

$$d(w_0, z_0) \leqslant d = d(\varepsilon)$$

with d constant, and such that

$$\left| \arg \frac{\alpha - z_0}{1 - \bar{z}_0 \alpha} - \arg \frac{\beta - z_0}{1 - \bar{z}_0 \beta} + \frac{\pi}{2} \right| < \varepsilon.$$

Because the δ in (4.8) can be chosen independent of $d(\varepsilon)$, Lemma 4.4 and Lemma 4.5 now yield

$$\left|\arg f'(z_0) - \arg\left(\frac{f(\beta)-f(z_0)}{1-\bar{f}(z_0)f(\beta)}\frac{1-\bar{z}_0\beta}{\beta-z_0}\right)\right| < \varepsilon.$$

Then from (4.9) we obtain

$$\left|\arg\frac{f(\beta)-f(z_0)}{1-\bar{f}(z_0)f(\beta)} - \arg\frac{f(\alpha)-f(z_0)}{1-\bar{f}(z_0)f_\alpha} - \frac{\pi}{2}\right| < 3\varepsilon,$$

and the geodesic $(f(z_0), f(b))$ is nearly a orthogonal to $(f(z_0, f(\alpha))$. Then by (4.7) we get

$$|f(\beta)-f(z_0)| < c_4(1-|f(z_0)|)$$

if c_4 is large enough and if ε is small. Consequently

$$|f'(\beta) \leqslant c_4 \frac{1-|f(z_I)|^2}{1-|z_I|^2},$$

and $\beta \in J \subset I$.

Conversely, assume that (4.1) holds. Let $S, T \in \mathcal{M}$, $S(0) = z_I$, $S(1) = \zeta$, $T(f(z_I)) = 0$, $T(f(\zeta)) = 1$, and set $g = T \circ f \circ S$. Then $g(0) = 0$, $g(1) = 1$ and for $z = S(t)$, $0 < t < 1$,

$$\frac{1-|g(t)|^2}{1-t^2} = \frac{(1-|f(z)|^2)(1-|f(z_I)|^2)|1-\bar{z}_I z|^2}{|1-\bar{f}(z_I)f(z)|^2(1-|z_I|^2)(1-|z|^2)} \leqslant c_5 A$$

by (4.1) and (3.7) since $S((0,1)) = (z_I, \zeta)$ lies inside a cone at ζ of fixed aperture and since

$$|1-\bar{f}(z_I)f(z)| \geqslant 1-|f(z_I)|.$$

Therefore, when $0 < t < x < 1$,

$$d(x,0) - d(g(x),0) = \log\frac{1+x}{1-x}\frac{1-|g(x)|}{1+|g(x)|} \leqslant \log(4c_5 A) = \delta$$

and

$$\begin{aligned} d(g(x), g(t) &\geqslant d(g(x),0) - d(0,g(t)) \\ &\geqslant d(x,0) - d(0,t) - \delta \\ &= d(x,t) - \delta. \end{aligned}$$

Therefore (3.4) holds in (z_I, ζ) with constant δ independent of I.

By Lemma 4.3 there is $z_0 \in (z_I, \zeta)$ such that, given $\eta > 0$, $d(z_0, z_I) \leqslant \rho(\eta)$ and

$$\frac{|f'(z_0)|(1-|z_0|^2)}{1-|f(z_0)|^2} \geqslant 1-\eta.$$

Let J_1 and J_2 be the two outer thirds of $I_0 = \{\zeta : |\zeta - z_0/|z_0|| < 1-|z_0|\}$. By hypothesis there is $\zeta_j \in K_j$ such that, for δ fixed, (3.4) holds on (z_I, ζ_j). If η is sufficiently small, then by Lemma 4.5 and Lemma 4.4,

$$\arg\left(\frac{f(\zeta_2)-f(z_0)}{1-\bar{f}(z_0)f(\zeta_2)}\left(\overline{\frac{f(\zeta_1)-f(\zeta_0)}{1-\bar{f}(z_0)f(\zeta_1)}}\right)\right) \geqslant \frac{\pi}{10}.$$

That means (3.2) holds (for a different δ) on the full geodesic $\sigma = (\zeta_1, \zeta_2)$. And clearly

$$\operatorname{dist}(z_I, \sigma) \leqslant \rho'(\eta) = \rho'(A).$$

Hence by Theorem 3.1, $f \in M(c)$ for $c = c(A)$.

5. *A condition on the zeros*

We have seen that every $f \in M(c)$ is a Blaschke product. Now suppose that f is a Blaschke product with zeros z_ν, $\nu = 1, 2, \ldots$. A theorem of Frostman (see [3, p. 177]) says that f has angular derivative at a point $\zeta \in \partial D$ if and only if

$$\sum_{\nu=1}^{+\infty} \frac{1-|z_\nu|^2}{|\zeta - z_\nu|^2} < +\infty$$

and in this case $|f'(\zeta)|$ is equal to this sum.

THEOREM 5.1. *Given $c > 0$ there is $A = A(c) < +\infty$ so that if f is a Blaschke product in $M(c)$ and $\{z_\nu\}$ are the zeros of f, then for every* arc $I \subset \partial D$ *with $|I| < \pi$,*

$$\inf_{\zeta \in I} \sum_\nu \frac{1-|z_\nu|^2}{|\zeta - z_\nu|^2} \leqslant A \sum_\nu \frac{1-|z_\nu|^2}{|1-\bar{z}_\nu z_I|^2} \tag{5.1}$$

and

$$(1-|z_I|^2) \sum_\nu \frac{1-|z_\nu|^2}{|1-\bar{z}_\nu z_I|^2} \leqslant A. \tag{5.2}$$

Conversely, given $A < +\infty$ there is $c = c(A) > 0$, so that if f is a Blaschke product with zeros $\{z_\nu\}$, such that (5.1) *and* (5.2) *are true for every* arc $I \subset \partial D$ *with $|I| < \pi$, then $f \in M(c)$.*

By [4, p. 286], condition (5.2) holds if and only if the measure

$$\sum (1-|z_\nu|)\, \delta z_\nu$$

is a Carleson measure with constant bounded by $C(A)$. That holds if and only if $\{z_\nu\}$ is the union of at most $N = N(A)$ interpolating sequences $\{z_j\}$ and

$$\delta(\{z_j\}) = \mathrm{int}_j \prod_{k,\, k \neq j} \left| \frac{z_k - z_j}{1 - \bar{z}_k z_j} \right| \geqslant \delta_0(A) > 0.$$

If $f(z)$ is the Blaschke product in the upper half-plane with zeros $\{n+i : n+\mathbb{Z}\}$ then f has (5.2) but by Lemma 2.1 f is in no $M(c)$ because $f(z) = \lambda(e^{2\pi i z} - e^{-2\pi})/(1-e^{-2\pi} e^{2\pi i z})$ with $|\lambda| = 1$ and

$$\lim_{y \to \infty} f(iy) = -\lambda e^{-2\pi}.$$

Proof. Suppose that $f \in M(c)$ and that $\{z_\nu\}$ is the zeros of f. Then by Theorem 4.1 there is $A = A(c) < \infty$ so that

$$\inf_{\zeta \in I} |f'(\zeta)| \leqslant A \frac{1-|f(z_I)|^2}{1-|z_I|^2}. \tag{5.3}$$

The above-mentioned theorem of Frostman says that

$$\inf_{z \in I} |f'(\zeta)| = \inf_{\zeta \in I} \sum_\nu \frac{1-|z_\nu|^2}{|\zeta - z_\nu|^2}.$$

Also

$$|f(z_I)|^2 = \prod_\nu \left|\frac{z_I - z_\nu}{1-\bar{z}_\nu z_I}\right|^2 \geqslant 1 - \sum_\nu \left(1 - \left|\frac{z_I - z_\nu}{1-\bar{z}_\nu z_I}\right|^2\right) = 1 - \sum_\nu \frac{(1-|z_I|^2)(1-|z_\nu|^2)}{|1-\bar{z}_\nu z_I|^2}.$$

Hence

$$\frac{1-|f(z_I)|^2}{1-|z_I|^2} \leqslant \sum_\nu \frac{1-|z_\nu|^2}{|1-\bar{z}_\nu z_I|^2}.$$

Therefore (5.3) implies (5.1). Now, there is an absolute constant K such that

$$|\zeta - z_\nu| \leqslant K|1-\bar{z}_\nu z_I|$$

for every $\zeta \in I$ and every z_ν. Then (5.3) implies that

$$\begin{aligned}\sum_\nu \frac{1-|z_\nu|^2}{|1-\bar{z}_\nu z_I|^2} &\leqslant K^2 \inf_{\zeta\in I} \sum_\nu \frac{1-|z_\nu|^2}{|\zeta - z_\nu|^2}\\ &\leqslant K^2 A \frac{1-|f(z_I)|^2}{1-|z_I|^2}\\ &\leqslant K^2 A \frac{1}{1-|z_I|^2}.\end{aligned}$$

Hence (5.2) is true.

Conversely suppose that (5.1) and (5.2) hold for every $I \subset \partial D$ with $|I| < \pi$. Then, if $|f(z_I)| \geqslant \frac{1}{2}$, there is an absolute constant K such that

$$\begin{aligned}K(1-|f(z_I)|^2) &\geqslant -\log|f(z_I)|^2 = -\log\left(\prod_\nu \left|\frac{z_I - z_\nu}{1-\bar{z}_\nu z_I}\right|^2\right)\\ &\geqslant \sum_\nu \left(1 - \left|\frac{z_I - z_\nu}{1-\bar{z}_\nu z_I}\right|^2\right).\end{aligned}$$

Hence

$$\sum_\nu \frac{1-|z_\nu|^2}{|1-\bar{z}_\nu z_I|^2} \leqslant K \frac{1-|f(z_I)|^2}{1-|z_I|^2}.$$

But, then (5.1) implies that

$$\inf_{\zeta\in I} |f'(\zeta)| \leqslant AK \frac{1-|f(z_I)|^2}{1-|z_I|^2}.$$

If $|f(z_I)| \leqslant \frac{1}{2}$, then (5.1) and (5.2) imply that

$$\inf_{\zeta\in I} |f'(\zeta)| \leqslant A \sum_\nu \frac{1-|z_\nu|^2}{|1-\bar{z}_\nu z_I|^2} \leqslant A^2 \frac{1}{1-|z_I|^2} \leqslant \tfrac{4}{3} A^2 \frac{1-|f(z_I)|^2}{1-|z_I|^2}.$$

So in both cases, $f \in M(c)$, with $c = c(A) > 0$, by Theorem 4.1.

6. *An example*

By Theorem 4.1 we know that if $f \in M(c)$ then f has angular derivative on a dense subset of ∂D. Lennart Carleson and Peter Jones found an example where the angular derivatives exist only on a set of measure zero. We thank them for letting us include it here.

THEOREM 6.1. *There exists a Blaschke product f such that $f \in M(c)$ for some $c > 0$, but f has no angular derivative outside a subset of ∂D of measure zero.*

Proof. Consider $\delta_k = k^{-2} \cdot 10^{-2k}, k \geqslant 1$ and

$$\theta_{j,k} = 2\pi j/10^k, \qquad 1 \leqslant j \leqslant 10^k, k \geqslant 1,$$
$$z_{j,k} = (1-\delta_k) e^{i\theta_{j,k}}.$$

Let f be the Blaschke product with zeros $\{z_{j,k}\}$. Set

$$I = [e^{i\theta_{j_0,n}}, e^{i\theta_{j_0+1,n}}], \qquad 1 \leqslant n, 0 \leqslant j_0 \leqslant 10^n - 1.$$

It suffices to prove (5.1) and (5.2) for such I.

From now on all the constants will be absolute constants and the same symbol may represent two or more constants. Now with $\theta_I = \arg(z_I)$

$$\begin{aligned}
(1-|z_I|^2)\sum_{j,k} \frac{1-|z_{j,k}|^2}{|1-\bar{z}_{j,k} z_I|^2} &\leqslant c10^{-n} \sum_{j,k} \frac{\delta_k}{(\theta_{j,k}-\theta_I)^2 + (\delta_k + |I|)^2} \\
&\leqslant c10^{-n} \sum_{\delta_k \leqslant 10^{-n}} \frac{\delta_k}{(\theta_{j,k}-\theta_I)^2 + 10^{-2n}} \\
&\quad + c10^{-n} \sum_{\delta_k > 10^{-n}} \frac{\delta_k}{(\theta_{j,k}-\theta_I)^2 + \delta_k^2} \\
&\leqslant c10^{-n} \sum_{\delta_k \leqslant 10^{-n}} \delta_k 10^k \int_T \frac{d\theta}{\theta^2 + 10^{-2n}} \\
&\quad + c10^{-n} \sum_{\delta_k > 10^{-n}} \delta_k 10^k \int_T \frac{d\theta}{\theta^2 + \delta_k^2} \\
&\leqslant c10^{-n} \sum_{\delta_k \leqslant 10^{-n}} \delta_k 10^k \int_{|\theta| \geqslant 10^{-n}} \frac{d\theta}{\theta^2} \\
&\quad + c10^{-n} \sum_{\delta_k \leqslant 10^{-n}} \delta_k 10^k \int_{|\theta| \leqslant 10^{-n}} \frac{d\theta}{10^{-2n}} \\
&\quad + c10^{-n} \sum_{\delta_k > 10^{-n}} \delta_k 10^k \int_{|\theta| \geqslant \delta_k} \frac{d\theta}{\theta^2} \\
&\quad + c10^{-n} \sum_{\delta_k > 10^{-n}} \delta_k 10^k \int_{|\theta| < \delta_k} \frac{d\theta}{\delta_k^2} \\
&= c10^{-n} \sum_{\delta_k \leqslant 10^{-n}} 10^n \delta_k 10^k + c10^{-n} \sum_{\delta_k \leqslant 10^{-n}} 10^n \delta_k 10^k \\
&\quad + c10^{-n} \sum_{\delta_k > 10^{-n}} 10^k + c^{-n} \sum_{\delta_k > 10^{-n}} 10^k \\
&= c \sum_{\delta_k \leqslant 10^{-n}} k^{-2} 10^{-k} + c10^{-n} \sum_{\delta_k > 10^{-n}} 10^k \leqslant c.
\end{aligned}$$

That proves (5.2).

Now choose θ so that $10^n(\theta/2\pi) = j_0 \cdot 444 \ldots$. Then $\zeta = e^{i\theta} \in I$ and moreover

$$|\theta - \theta_{j,k}| \geqslant \begin{cases} c10^{-k} & \text{if } k \geqslant n, \\ c10^{-n} & \text{if } k < n. \end{cases} \tag{6.1}$$

Denote by $\theta_k^*, k \geqslant 1$, that $\theta_{j,k}$ which is closest to θ. Then

$$\sum_{j,k} \frac{1-|z_{j,k}|^2}{|\zeta - z_{j,k}|^2} = \sum_k \sum_{z_{j,k} \neq z_k^*} \frac{1-|z_{j,k}|^2}{|\zeta - z_{j,k}|^2} + \sum_k \frac{1-|z_k^*|^2}{|\zeta - z_k^*|^2} = \mathrm{I} + \mathrm{II}.$$

But

$$\begin{aligned} \mathrm{I} &\leqslant c \sum_k \delta_k \sum_{z_{j,k} \neq z_k^*} \frac{1}{(\theta - \theta_{j,k})^2 + \delta_k^2} \\ &\leqslant c \sum_k \delta_k 10^k \int_{|\theta| \geqslant 10^{-k}} \frac{d\theta}{\theta^2 + \delta_k^2} \\ &\leqslant c \sum_k \delta_k 10^{2k} = c \sum_k k^{-2} \leqslant c. \end{aligned}$$

Therefore,

$$\mathrm{I} \leqslant c \sum_{j,k} \frac{1-|z_{j,k}|^2}{|1 - \bar{z}_{j,k} z_I|^2}$$

and to prove (5.1), it remains only to prove that

$$\mathrm{II} \leqslant c \sum_{j,k} \frac{1-|z_{j,k}|^2}{|1 - z_{j,k} z_I|^2}.$$

But by (6.1),

$$\begin{aligned} \mathrm{II} &\leqslant c \sum_k \frac{\delta_k}{(\theta - \theta_k^*)^2 + \delta_k^2} = c \sum_{k \geqslant n} \frac{\delta_k}{(\theta - \theta_k^*)^2 + \delta_k^2} + c \sum_{k<n} \frac{\delta_k}{(\theta - \theta_k^*)^2 + \delta_k^2} \\ &\leqslant c \sum_{k \geqslant n} \frac{\delta_k}{10^{-2k} + \delta_k^2} + c \sum_{k<n} \frac{\delta_k}{(\theta - \theta_k^*)^2 + \delta_k^2} \leqslant c + c \sum_{k<n} \frac{\delta_k}{(\theta - \theta_k^*)^2 + \delta_k^2}. \end{aligned}$$

Now if $k < n$ then

$$|\theta - \theta_k^*|^2 \geqslant c(|\theta - \theta_I|^2 + (\theta_I - \theta_k^*)^2) \geqslant c(|I|^2 + (\theta_I - \theta_k^*)^2).$$

Therefore

$$\begin{aligned} \mathrm{II} &\leqslant c + c \sum_{k<n} \frac{\delta_k}{(\theta_I - \theta_k^*)^2 + |I|^2 + \delta_k^2} \\ &\leqslant c + c \sum_{j,k} \frac{\delta_k}{(\theta_I - \theta_{j,k})^2 + (|I| + \delta_k)^2} \\ &\leqslant c + c \sum_{j,k} \frac{1-|z_{j,k}|^2}{|1 - \bar{z}_{j,k} z_I|^2}. \end{aligned}$$

Altogether

$$\sum_{j,k} \frac{1-|z_{j,k}|^2}{|\zeta - z_{j,k}|^2} = \mathrm{I} + \mathrm{II} \leqslant c \sum_{j,k} \frac{1-|z_{j,k}|^2}{|1 - \bar{z}_{j,k} z_I|^2}$$

and we obtain (5.1). So f satisfies (5.1) and (5.2), and by Theorem 5.1, $f \in M(c)$ for some $c > 0$.

Now let $E_{j,k} = [\exp(i\theta_{j,k}), \exp(i(\theta_{j,k}+2\pi\sqrt{\delta_k}))]$ and $E_k = \bigcup_j E_{j,k}$. Then

$$c\sqrt{\delta_k} \leqslant |E_{j,k}| \leqslant c\sqrt{\delta_k} \quad \text{and} \quad c(1/k) \leqslant |E_k| \leqslant c(1/k).$$

If $\zeta \in \limsup E_k$, then ζ belongs to infinitely many $E_{j,k}$. Therefore

$$\frac{1-|z_{j,k}|^2}{|\zeta - z_{j,k}|^2} \geqslant c\frac{\delta_k}{(\theta-\theta_{j,k})^2+\delta_k^2} \geqslant c$$

for infinitely many (j,k). Hence

$$\sum_{j,k} \frac{1-|z_{j,k}|^2}{|\zeta - z_{j,k}|^2} = +\infty$$

and f has no angular derivative at ζ. It remains to prove that $\limsup E_k$ has full measure in ∂D. Consider, instead,

$$\tilde{E}_k = \left\{e^{i\theta} : \frac{\theta}{2\pi} = 0\cdot x_1 x_2 \ldots \text{ and } x_n = 0, n = k, k+1, \ldots, k+[\log_{10} k]\right\}.$$

Then $\tilde{E}_k \subset E_k$, and it is enough to prove that $\limsup \tilde{E}_k$ has full measure in ∂D. The idea is that the sets $\tilde{E}_k$ act like independent events.

Claim 1. $\sum_k |\tilde{E}_k| = +\infty$.

Proof. This is clear since $|\tilde{E}_k| \geqslant c/k, k \geqslant 1$.

Claim 2. *There is c so that if $m < n$,*

$$|\tilde{E}_m \cap \tilde{E}_n| \leqslant c|\tilde{E}_m|\,|\tilde{E}_{n-m}|.$$

Proof. *Case* 1: $n > m+[\log_{10} m]$. In this case

$$|\tilde{E}_m \cap \tilde{E}_n| = c\frac{1}{n}\frac{1}{m} < \frac{1}{m}\frac{1}{n-m} \leqslant c|\tilde{E}_m|\,|\tilde{E}_{n-m}|.$$

Case 2: $m < n < m+[\log_{10} m]$. Then

$$\tilde{E}_m \cap \tilde{E}_n = \left\{e^{i\theta} : \frac{\theta}{2\pi} = 0\cdot x_1 x_2 \ldots \text{ and } x_k = 0, k = m, m+1, \ldots, n+[\log_{10} n]\right\}.$$

Hence

$$|\tilde{E}_m \cap \tilde{E}_n| = c\left(\frac{1}{10}\right)^{n+[\log_{10} n]-m} = c\frac{1}{n}10^{m-n} \leqslant c\frac{1}{m}\frac{1}{n-m} \leqslant c|\tilde{E}_m|\,|\tilde{E}_{n-m}|.$$

Now by Claim 1 and Claim 2 and by [**2**, Exercise 18 p. 79]

$$|\limsup \tilde{E}_k| > 0.$$

Moreover, $\tilde{E} = \limsup \tilde{E}_k$ is invariant under translation (that is, rotation) by any $e^{i\theta_{j,k}}$. Because these points are dense on the circle, a point of density argument shows $\tilde{E}$ has full measure.

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