One sided extendability and p-continuous analytic capacities

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Dedicated to Professor Jean-Pierre Kahane on the occasion of his 90th birthday.

Abstract

Using complex methods combined with Baire's Theorem we show that one-sided extendability, extendability and real analyticity are rare phenomena on various spaces of functions in the topological sense. These considerations led us to introduce the p-continuous analytic capacity and variants of it, $p \in \{0, 1, 2, \cdots\} \cup \{\infty\}$, for compact or closed sets in $\mathbb{C}$. We use these capacities in order to characterize the removability of singularities of functions in the spaces A^p.

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1 Introduction

In [2] it is proven that the set X of nowhere analytic functions in $C^\infty([0,1])$ contains a dense and G_δ subset of $C^\infty([0,1])$. In [1] using Fourier methods it is shown that X is itself a dense and G_δ subset of $C^\infty([0,1])$. Furthermore, combining the above methods with Borel's Theorem ([7]) and a version of Michael's Selection Theorem ([12]) the above result has been extended to $C^\infty(\gamma)$, where γ is any analytic curve. In the case where γ is the unit circle T every function $f \in C^\infty(T)$ can be written as a sum $f = g + w$ where

g belongs to $A^\infty(D)$ and is holomorphic in the open unit disc D and very smooth up to the boundary and w has similar properties in D^c. Now, if we assume that f is extendable somewhere towards one side of T, say in D^c, then because w is regular there, it follows that $g \in A^\infty(D)$ is extendable. But the phenomenon of somewhere extendability has been proven to be a rare phenomenon in the Frechet space $A^\infty(D)$ ([8]). It follows that the phenomenon of one sided somewhere extendability is a rare phenomenon in $C^\infty(T)$ or more generally in $C^p(\gamma)$, $p \in \{\infty\} \cup \{0, 1, 2, \cdots\}$ for any analytic curve γ ([1]).

After the preprint [1] has been circulated, P. Gautier noticed that the previous result holds more generally for Jordan arcs without the assumption of analyticity of the curve. Indeed, applying complex methods, appearing in the last section of [1], we prove this result. It suffices to use the Oswood-Caratheodory Theorem combined with Montel's Theorem and the Poisson integral formula applied to the boundary values of bounded holomorphic functions in $H^\infty(D)$. In fact, this complex method is most natural to our considerations of extendability, real-analyticity and one sided-extendability. The proofs are simplified and the results hold under much more general assumptions than the assumptions imposed by the Fourier method. This complex method is developed in the present paper.

In section 4 we prove that extandability and real analyticity are rare phenomena in various spaces of functions on locally injective curves γ. For the real analyticity result we assume that γ is analytic and the result holds in any $C^k(\gamma), k \in \{0, 1, 2, 3, \cdots\} \cup \{\infty\}$ endowed with its nature topology. For the other results the phenomena are proven to be rare in $C^k(\gamma)$, provided that the locally injective curve γ has smoothness at least of degree k.

In section 5 initially we consider a finite set of disjoint curves $\gamma_1, \gamma_2, \cdots, \gamma_n$. Then in the case where $\gamma_1, \gamma_2, \cdots, \gamma_n$ are disjoint Jordan curves in $\mathbb{C}$ bounding a domain Ω of finite connectivity we consider the spaces $A^p(\Omega), p \in \{0, 1, 2, 3, \cdots\} \cup \{\infty\}$ which by the maximum principle can be seen as function spaces on $\partial\Omega = \gamma_1^* \cup \cdots \cup \gamma_n^*$. In these spaces we show that the above phenomena of extendability or real analyticity are rare phenomena. For the real analyticity result we assume analyticity of $\partial\Omega$, but for the extendability result we do not need to assume any smoothness of the boundary.

In section 6 we consider the one sided extendability from a locally injective curve γ and we prove that this is a rare phenomenon in various spaces of functions. We construct a denumerable family G_n of Jordan domains G containing in their boundary a non-trivial arc J of the image γ^* of γ, such that each other domain Ω with similar properties contains some G_n. We show that the phenomenon of extendability is rare for each domain G. Then by denumerable union (or intersection of the complements) we obtain our result with the aid of Baire's Category Theorem. We mention that the one-sided extendability of a function $f : \gamma^* \to \mathbb{C}$ is meant as the existence of a function $F : G \cup J \to \mathbb{C}$ which is hololomorphic on the Jordan domain G, continuous on $G \cup J$ and such that on the arc J of γ^* we have $F|_J = f|_J$. Such notions of one-sided extendability have been considered in [3] and the references therein, but in the present article and [1] it is, as far as we know, the first time where the phenomenon is proven to be rare.

At the end of section 6 we prove similar results on one-sided extendability on the space $A^p(\Omega)$, where Ω is a finitely connected domain in $\mathbb{C}$ bounded by a finite set of disjoint Jordan curves $\gamma_1, \gamma_2, \cdots, \gamma_n$. Now, the extension F of a function $f \in A^p(\Omega)$ has to coincide with f only on a non-trivial arc of the boundary of Ω, not on an open subset of Ω. Certainly if the continuous analytic capacity of $\partial\Omega$ is zero, the latter automatically happens, but not in general.

In section 7 we consider Ω a domain in $\mathbb{C}$, L a compact subset of Ω and we consider the phenomenon of extendability of a function $f \in A^p(\Omega \setminus L)$ to a function F in $A^p(\Omega)$. There is a dichotomy. Either for every f this is possible or generically for all $f \in A^p(\Omega \setminus L)$ this fails. In order to characterize when each horn of the above dichotomy holds we are led to define the p-continuous analytic capacity $a_p(L)$ ($p \in \{0, 1, 2, 3, \cdots\} \cup \{\infty\}$), where $a_0(L)$ is the known continuous analytic capacity $a_0(L) = a(L)$ ([5]).

The study of the above capacities and variants of it is done in section 3. For $p = 1$ the p-continuous analytic capacity a_1 is distinct from the continuous analytic capacity $a_0 = a$. In particular, if $K_{1/3}$ is the usual Cantor set lying on $[0, 1]$ and $L = K_{1/3} \times K_{1/3}$, then $a_0(L) > 0$, but $a_1(L) = 0$. This means that for any open set U containing L there exists a function in $A(U \setminus L)$ which is not holomorhic in U, but if the derivative of a function in $A(U \setminus L)$ extends continuously on L, then the function is holomorphic in U. Generic versions of this fact imply that $A^1(U \setminus L)$ is of first category in $A^0(U \setminus L) = A(U \setminus L)$. If we replace the spaces A^p with the $\tilde{A}^p$ spaces of Whitney type, then the extension on L is equivalent to the fact that the interior of L is void. Thus we can define the continuous analytic capacities $\tilde{a}^p(L)$ but they vanish if and only if the interior of L is empty. We prove what is needed in section 7. More detailed study of those capacities will be done in future papers; for instance we can investigate the semiadditivity of a_p, whether the vanishing of a_p on a

compact set L is a local phenomenon and whether replacing the continuous analytic capacity a by the Ahlfors analytic capacity γ we can define capacities γ_p satisfying the analogous properties. Certainly the spaces $A^p(\Omega)$ will be replaced by $H_p^\infty(\Omega)$, the space of holomorphic functions on Ω such that for every $l \in \mathbb{N}$, $l \leq p$ the derivative $f^{(l)}$ of order l is bounded on Ω. We will also examine if a dichotomy result as in section 7 holds for the spaces $H_p^\infty(\Omega)$ in the place of $A^p(\Omega)$. All these in future papers.
In section 2 some preliminary geometry of locally injective curves is presented; for instance a curve is real analytic if and only if real analyticity of a function on the curve is equivalent to holomorphic extendability of the function on discs centred on points of the curve. We also show that if the map γ defining the curve is a homeomorphism with non-vanishing derivative, then the spaces $C^k(\gamma), k \in \{0, 1, 2, 3, \cdots\} \cup \{\infty\}$ are independent of the particular parametrization γ and depend only on the image γ^* of γ. Thus, in some cases it makes sense to write $C^k(\partial\Omega)$ and prove generic results in these spaces, $k \in \{0, 1, 2, 3, \cdots\} \cup \{\infty\}$.
Finally, we mention that some of the results of section 5 are valid for analytic curves γ; that is, they hold when we use a conformal parametrization of γ. Naturally comes the question whether these results remain true if we change the parametrization of the curve; in particular what happens if we consider the parametrization with respect to the arc length s? Answering this question was the motivation of [9], [10] where it is proven that arc length is a global conformal parameter for any analytic curve. Thus, the results of section 5 remain true if we use the arc length parametrization. Finally, we mention that in the present paper we start with qualitative categorical results, which lead us to quantitative notions as the p-continuous analytic capacity a_p and the p-analytic capacity γ_p.

2 Preliminaries

In most of our results it is important what is the degree of smoothness of a curve and the relation of real analyticity of functions on a curve with the holomorphic extendability of them around the curve. That is why we present here some basic results concerning locally injective curves in $\mathbb{C}$.

Definition 2.1. *Let $\gamma : I \to \mathbb{C}$ be a continuous and locally injective function, where I is an interval or the unit circle. Let also $l \in \{1, 2, 3, \cdots\} \cup \{\infty\}$. The curve γ belongs to the class $C^l(I)$, if for $k \in \{1, 2, 3, \cdots\}, k \leq l$, the* *derivative $\gamma^{(k)}$ exists and is a continuous function.*

Definition 2.2. *Let $\gamma : I \to \mathbb{C}$ be a continuous and locally injective function, where I is an interval or the unit circle. Let also $k \in \{1, 2, 3, \cdots\} \cup \{\infty\}$. A function $f : \gamma^* \to \mathbb{C}$ defined on the image $\gamma^* = \gamma(I)$ belongs to the class $C^k(\gamma)$, if for $l \in \{1, 2, 3, \cdots\}, l \leq k$, the derivative $(f \circ \gamma)^l$ exists and is a continuous function. Let $(I_n), n \in \{1, 2, 3, \cdots\}$ be an increasing sequence of compact intervals such that $\bigcup_{n=1}^{\infty} I_n = I$. The topology of the space $C^k(\gamma)$ is defined by the seminorms*

$$\sup_{t \in I_n} |(f o \gamma)^{(l)}(t)|, l = 0, 1, 2, ..., k, n = 1, 2, 3, \ldots$$

In this way $C^k(\gamma)$, $k = 0, 1, 2, ...$ becomes a Banach space if I is compact , $C^k(\gamma)$, $k = 0, 1, 2, ...$ a Frechet space if I is not compact and $C^\infty(\gamma)$ a Frechet space. Therefore Baire's theorem is at our disposal.

Definition 2.3. *Let $\gamma : I \to \mathbb{C}$ be a continuous and locally injective function, where I is an interval. We will say that the curve γ is analytic at $t_0 \in I$ if there exist an open set $t_0 \in V \subseteq \mathbb{C}$, a real number $\delta > 0$ with $(t_0 - \delta, t_0 + \delta) \cap I \subset V$ and a holomorphic and injective function $F : V \to \mathbb{C}$ such that $F|_{(t_0-\delta, t_0+\delta)} = \gamma|_{(t_0-\delta, t_0+\delta)}$. If γ is analytic at every $t \in I$, we will say that γ is an analytic curve.*

Lemma 2.4. *Let $t_0 \in I$ and $\gamma : I \to \mathbb{C}$ be a continuous and locally injective function, where I is an interval.*
For every function $f : I \to \mathbb{C}$ we suppose that 1) and 2) are equivalent:
1) There exists a power series of real variable

$$\sum_{n=0}^{\infty} a_n (t - t_0)^n, a_n \in \mathbb{C}$$

with a positive radius of convergence $r > 0$ and there exists $0 < \delta \leq r$ such that

$$f(t) = \sum_{n=0}^{\infty} a_n (t - t_0)^n$$

for $t \in (t_0 - \delta, t_0 + \delta) \cap I$.
2) There exists a power series of complex variable

$$\sum_{n=0}^{\infty} b_n (z - \gamma(t))^n, b_n \in \mathbb{C}$$

with a positive radius of convergence $s > 0$ and $0 < \epsilon \leq s$ such that

$$f(t) = \sum_{n=0}^{\infty} b_n(\gamma(t) - \gamma(t_0))^n$$

for $t \in (t_0 - \epsilon, t_0 + \epsilon) \cap I$.
Then γ is analytic at t_0.

Proof. We will use the implication 2) $\Rightarrow$ 1) only to prove that γ is differentiable at an open interval which contains t_0. We consider $\beta > 0$ and $J = (t_0 - \beta, t_0 + \beta) \cap I$. For every $t \in J$ we choose $f(t) = \gamma(t) = \gamma(t_0) + (\gamma(t) - \gamma(t_0))$ and so by the 2) $\Rightarrow$ 1) we have that there exists $0 < \delta < \beta$ such that

$$\gamma(t) = \sum_{n=0}^{\infty} a_n(t - t_0)^n$$

for some constants $a_n \in \mathbb{C}$ for every $t \in (t_0 - \delta, t_0 + \delta) \cap I$. Therefore γ is differentiable in this interval. Now, for $g(t) = t = t_0 + (t - t_0)$ by 1) $\Rightarrow$ 2) we have that there exists $0 < \epsilon \leq \delta$ such that

$$t = \sum_{n=0}^{\infty} b_n(\gamma(t) - \gamma(t_0))^n$$

for some constants $b_n \in \mathbb{C}$ for every $t \in (t_0 - \epsilon, t_0 + \epsilon)$. We differentiate the above equation at $t = t_0$ and we have that $1 = b_1\gamma'(t_0)$. Therefore $b_1 \neq 0$. The power series $\sum_{n=0}^{\infty} b_n(z - \gamma(t_0))^n$ has a positive radius of convergence and so there exists $\alpha > 0$ such that $\gamma(t) \in D(\gamma(t_0), \alpha)$ for every $t \in (t_0 - \epsilon, t_0 + \epsilon) \cap I$ and the function $f : D(\gamma(t_0), \alpha) \to \mathbb{C}$ with

$$f(z) = \sum_{n=0}^{\infty} b_n(z - \gamma(t_0))^n$$

is a holomorphic one. Also, we have that $f(\gamma(t)) = t$ for every $t \in (t_0 - \epsilon, t_0 + \epsilon) \cap I$. Because $f'(\gamma(t_0)) = b_1 \neq 0$, f is locally invertible and let $h = f^{-1}$ in an open disk $D(t_0, \eta)$ where $0 < \eta < \epsilon$. Then, $\gamma(t) = h(t)$ for every $t \in (t_0 - \eta, t_0 + \eta) \cap I$ and h is holomorphic and injective and the proof is complete. $\square$

Remark 2.5. The above proof shows that if γ in Lemma 2.4 belongs to $C^1(I)$, then the conclusion of the lemma is true even if we only assume that 1) $\Rightarrow$ 2) is true.

The following lemma is the inverse of Lemma 2.4.

Lemma 2.6. *Let $t_0 \in I$ and $\gamma : I \to \mathbb{C}$ be a continuous and locally injective function, which is analytic at t_0, where I is an interval. Let also $f : I \to \mathbb{C}$. Then the followings are equivalent:*

1) There exists a power series of real variable

$$\sum_{n=0}^{\infty} a_n (t - t_0)^n, a_n \in \mathbb{C}$$

with positive radius of convergence $r > 0$ and there exists a $0 < \delta \leq r$ such that

$$f(t) = \sum_{n=0}^{\infty} a_n (t - t_0)^n$$

for $t \in (t_0 - \delta, t_0 + \delta) \cap I$.

2) There exists a power series with complex variable

$$\sum_{n=0}^{\infty} b_n (z - \gamma(t_0))^n, b_n \in \mathbb{C}$$

with positive radius of convergence $s > 0$ and there exists $\epsilon > 0$ such that

$$f(t) = \sum_{n=0}^{\infty} b_n (\gamma(t) - \gamma(t_0))^n$$

for $t \in (t_0 - \epsilon, t_0 + \epsilon) \cap I$.

Proof. Because γ is an analytic curve at t_0, there exists an open disk $D(t_0, \varepsilon) \subseteq \mathbb{C}$, where $\varepsilon > 0$ and a holomorphic and injective function $\Gamma : D(t_0, \varepsilon) \to \mathbb{C}$ with $\Gamma(t) = \gamma(t)$ for $t \in (t_0 - \varepsilon, t_0 + \varepsilon) \cap I$.

$(i) \Rightarrow (ii)$ We consider the function

$$g(z) = \sum_{n=0}^{\infty} a_n (z - t_0)^n,$$

$z \in D(t_0, \delta)$, which is well defined and holomorphic in $D(t_0, \delta)$. We have that

$$\Gamma^{-1} : \Gamma(D(t_0, \varepsilon)) \to \mathbb{C}$$

is a holomorphic function. We consider the function

$$F = go\Gamma^{-1} : \Gamma(D(t_0, \varepsilon)) \to \mathbb{C},$$

(where $\Gamma(D(t_0, \varepsilon))$ is an open set) which is a holomorphic function and

$$(Fo\Gamma)(t) = f(t), t \in D(t_0, \varepsilon) \cap I.$$

Therefore, there exist $b_n \in \mathbb{C}$,$n = 1, 2, 3, ...$ and $\delta > 0$ such that

$$F(z) = \sum_{n=0}^{\infty} b_n (z - \gamma(t_0))^n$$

for every $z \in D(\gamma(t_0), \delta) \subseteq \Gamma(D(t_0, \varepsilon))$ and thus

$$f(t) = (Fo\gamma)(t) = \sum_{n=0}^{\infty} b_n (\gamma(t) - \gamma(t_0))^n$$

in an interval $(t_0 - s, t_0 + s) \cap I$ where $s > 0$.

$(ii) \Rightarrow (i)$ We consider the function

$$G(z) = \sum_{n=0}^{\infty} b_n (z - \gamma(t_0))^n,$$

$z \in D(\gamma(t_0), s)$. We choose $a > 0$ with $a < \varepsilon$ such that $\Gamma(D(t_0, a)) \subseteq D(\gamma(t_0), s)$. The function

$$Go\Gamma : D(t_0, a) \to \mathbb{C}$$

is holomorphic. Therefore, there exist $a_n \in \mathbb{C}$,$n = 1, 2, 3, ...$ such that

$$(Go\Gamma)(z) = \sum_{n=0}^{\infty} a_n (z - t_0)^n, z \in D(t_0, a)$$

and consequently

$$f(t) = G(\gamma(t)) = \sum_{n=0}^{\infty} a_n (t - t_0)^n, t \in (t_0 - a, t_0 + a).$$

□

Definition 2.7. *Let $\gamma : I \to \mathbb{C}$ be a locally injective curve and $z_0 = \gamma(t_0), t_0 \in I$, where I is an interval. A function $f : \gamma^* \to \mathbb{C}$ belongs to the class of non-holomorphically extendable at $(t_0, z_0 = \gamma(t_0))$ functions, if there are no open disk $D(z_0, r), r > 0$ and $\eta > 0$ and a holomorphic function $F : D(z_0, r) \to \mathbb{C}$, such that $\gamma((t_0 - \eta, t_0 + \eta) \cap I) \subset D(z_0, r)$ and $F(\gamma(t)) = f(\gamma(t))$ for all $t \in (t_0 - \eta, t_0 + \eta) \cap I$. Otherwise we say that f is holomorphically extendable at $(t_0, z_0 = \gamma(t_0))$.*

Definition 2.8. *Let $\gamma : I \to \mathbb{C}$ be a continuous map, where I is an interval and $t_0 \in I$. A function $f : \gamma^* \to \mathbb{C}$ is real analytic at $(t_0, z_0 = \gamma(t_0))$, if there exist $\delta > 0$ and a power series $\sum\limits_{n=0}^{\infty} a_n (t - t_0)^n$ with a radius of convergence $\epsilon > \delta > 0$, such that $f(\gamma(t)) = \sum\limits_{n=0}^{\infty} a_n (t - t_0)^n$ for every $t \in (t_0 - \delta, t_0 + \delta) \cap I$.*

The following proposition associates the phenomenon of real-analyticity and that of holomorphically extendability.

Proposition 2.9. *Let $\gamma : I \to \mathbb{C}$ be an analytic curve at t_0, where I is an interval and $t_0 \in I$. A function $f : \gamma^* \to \mathbb{C}$ is real analytic at $(t_0, z_0 = \gamma(t_0))$ if and only if f is holomorphically extendable at $(t_0, z_0 = \gamma(t_0))$.*

Proof. At first, we will prove direction $\Rightarrow$: If f is real analytic at $(t_0, z_0 = \gamma(t_0))$, then from Lemma 2.6

$$f(\gamma(t)) = \sum_{n=0}^{\infty} b_n (\gamma(t) - \gamma(t_0))^n$$

for every $t \in (t_0-\epsilon, t_0+\epsilon)\cap I$ and for some $b_n \in \mathbb{C}, \epsilon > 0$. From the continuity of γ, there exists $\eta > 0$ such that $\gamma((t_0-\eta, t_0+\eta)\cap I) \subset D(z_0, \epsilon)$. Therefore, the function

$$F(z) = \sum_{n=0}^{\infty} b_n(z-\gamma(t_0))^n$$

for $z \in D(z_0, \epsilon)$ is equal to f on $\gamma((t_0-\eta, t_0+\eta)\cap I)$. Thus the function f is holomorphically extendable at $(t_0, z_0 = \gamma(t_0))$.
Next we prove direction $\Leftarrow$: If f is extendable at $(t_0, z_0 = \gamma(t_0))$, then there exist $r > 0$ and a holomorphic function $F : D(\gamma(t_0), r) \to \mathbb{C}$, such that

$$f(\gamma(t)) = F(\gamma(t))$$

for every $t \in (t_0-\epsilon, t_0+\epsilon)\cap I$ and for some $\epsilon > 0$. Let

$$\sum_{n=0}^{\infty} b_n(z-\gamma(t_0))^n$$

be the Taylor expansion of the holomorphic function F. It follows that

$$f(\gamma(t)) = F(\gamma(t)) = \sum_{n=0}^{\infty} b_n(\gamma(t)-\gamma(t_0))^n$$

for every $t \in (t_0-\epsilon, t_0+\epsilon)\cap I$ and as a result, again from Lemma 2.6, we conclude that f is real analytic at $(t_0, z_0 = \gamma(t_0))$, because the curve γ is analytic at t_0. $\square$

The following theorem is a consequence of Lemma 2.4 and Proposition 2.9.

Theorem 2.10. *Let $\gamma : I \to \mathbb{C}$ be a continuous and locally injective curve, where I is an interval and $t_0 \in I$. Then, γ is analytic at t_0 if and only if for every function $f : \gamma^* \to \mathbb{C}$ the following are equivalent:*
1) f is real analytic at $(t_0, z_0 = \gamma(t_0))$
2) f is holomorphically extendable at $(t_0, z_0 = \gamma(t_0))$

Now, we will examine a different kind of differentiability.

Definition 2.11. *Let $\gamma : I \to \mathbb{C}$ be a continuous and injective curve, where I is an interval or the unit circle. We define the derivative of a function $f : \gamma^* \to \mathbb{C}$ at $\gamma(t_0)$, where $t_0 \in I$ by*

$$\frac{df}{dz}(\gamma(t_0)) = \lim_{t \to t_0} \frac{f(\gamma(t)) - f(\gamma(t_0))}{\gamma(t) - \gamma(t_0)}$$

if the above limit exists and is a complex number.

Remark 2.12. If $\gamma : I \to \mathbb{C}$ is a hemeomorphism from I to γ^*, then we can equivalently define $\frac{df}{dz}(\gamma(t_0))$ as

$$\frac{df}{dz}(\gamma(t_0)) = \lim_{z \in \gamma^*, z \to \gamma(t_0)} \frac{f(z) - f(\gamma(t_0))}{z - \gamma(t_0)}$$

if the above limit exists and is a complex number.

Definition 2.13. *Let $\gamma : I \to \mathbb{C}$ be a homeomorphism, where I is an interval or the unit circle. A function $f : \gamma^* \to \mathbb{C}$ belongs to the class $C^1(\gamma^*)$ if $\frac{df}{dz}(\gamma(t))$ exists and is continuous for $t \in I$. Also, for $k \in \{2, 3, \cdots\} \cup \{\infty\}$ a function $f : \gamma^* \to \mathbb{C}$ belongs to the class $C^k(\gamma^*)$, if*

$$\frac{d^k f}{dz^k}(\gamma(t)) = \frac{d\left(\frac{d^{k-1} f}{dz^{k-1}}(\gamma(t))\right)}{dz}(\gamma(t))$$

exists and is continuous for $t \in [0, 1]$. Finally, a function $f : \gamma^* \to \mathbb{C}$ belongs to the class $C^\infty(\gamma^*)$, if $\frac{d^k f}{dz^k}(\gamma(t))$ exists and is continuous for $t \in I$ and for every $k \in \{1, 2, 3, \cdots\}$. Let also $(I_n), n \in \{1, 2, 3, \cdots\}$ be an increasing sequence of compact subsets of γ^* such that $\bigcup_{n=1}^{\infty} I_n = \gamma^*$. For $k \in \{1, 2, 3, \cdots\} \cup \{\infty\}$ the topology of the space $C^k(\gamma^*)$ is defined by the seminorms

$$\sup_{z \in I_n} \left|\frac{d^{(l)} f}{dz^l}(z)\right|, l = 0, 1, 2, ..., k, n = 1, 2, 3, \ldots$$

In this way $C^k(\gamma^*)$ becomes a Banach space if $k < \infty$ and I is compact. Otherwise, $C^k(\gamma^*)$ is a Frechet space.

Proposition 2.14. *Let $\gamma : I \to \mathbb{C}$ be a homeomorphism with $\gamma'(t) \neq 0$ for all $t \in I$, where I is an interval or the unit circle and $k \in \{1, 2, 3, \cdots\} \cup \{\infty\}$. If $C^k(I) = C^k(\gamma^*) \circ \gamma$, then $\gamma \in C^k(I)$.*

Proof. The function $f : \gamma^* \to \mathbb{C}$ with $f(\gamma(t)) = \gamma(t)$ for $t \in I$ belongs to the class $C^k(\gamma^*)$ and therefore the function $\gamma = f \circ \gamma : I \to \mathbb{C}$ belongs to the class $C^k(I)$. $\square$

Now, we will prove the inverse of the previous proposition: If $\gamma \in C^k(I)$, then $C^k(I) = C^k(\gamma^*) \circ \gamma$. In order to do that we need the following lemma, which will also be useful later.

Lemma 2.15. *Let X be an interval $I \subset \mathbb{R}$ or the unit circle T, $\gamma \in C^k(X)$, $k \in \{1, 2, ...\} \cup \{\infty\}$ and $f \in C^k(\gamma^*)$ and $g = f \circ \gamma$. Then $g \in C^k(X)$ and there exist polynomials $P_{j,i}$, $i, j \in \{1, 2, 3, ...\}$, $j \leq i \leq k$, defined on $\mathbb{C}^i$, such that*

$$g^{(i)}(t) = \sum_{j=1}^{i} \frac{d^j f}{dz^j}(\gamma(t)) P_{j,i}(\gamma^{'}(t), \gamma^{''}(t), ..., \gamma^{(i)}(t))$$

where the derivatives of γ are with respect to the real variable t in the case $X = I$ and with respect to the complex variable t, $|t| = 1$ in the case $X = T$.

Proof. We will prove the lemma by induction on i. For $i = 1 \leq k$,

$$g^{'}(t) = (f \circ \gamma)^{'}(t) = \lim_{t \to t_0} \frac{f(\gamma(t)) - f(\gamma(t_0))}{t - t_0}$$

$$= \lim_{t \to t_0} \frac{f(\gamma(t)) - f(\gamma(t_0))}{\gamma(t) - \gamma(t_0)} \frac{\gamma(t) - \gamma(t_0)}{t - t_0} = \frac{df}{dz}(\gamma(t))\gamma^{'}(t)$$

and thus $g \in C^1(X)$ and $P_{1,1}(z) = z$, $z \in \mathbb{C}$.

If the result holds for $1 \leq i < k$, then we will prove that it also holds for $i + 1 \leq k$. Using our induction hypothesis,

$$g^{(i)}(t) = \sum_{j=1}^{i} \frac{d^j f}{dz^j}(\gamma(t)) P_{j,i}(\gamma^{'}(t), \gamma^{''}(t), ..., \gamma^{(i)}(t)).$$

We differentiate with respect to $t \in X$. Then

$$g^{(i+1)}(t) = \sum_{j=1}^{i} (\frac{d^{j+1} f}{dz^{j+1}}(\gamma(t))(P_{j,i}(\gamma^{'}(t), \gamma^{''}(t), ..., \gamma^{(i)}(t))\gamma^{'}(t)$$

$$+\frac{d^j f}{dz^j}(\gamma(t))\sum_{s=1}^{i}\frac{\partial P_{j,i}}{\partial z_s}(\gamma^{'}(t),\gamma^{''}(t),...,\gamma^{(i)}(t))\gamma^{(s+1)}(t)) =$$

$$= \frac{df}{dz}(\gamma(t))\sum_{s=1}^{i}\frac{\partial P_{1,i}}{\partial z_s}(\gamma^{'}(t),\gamma^{''}(t),...,\gamma^{(i)}(t))\gamma^{s+1}(t)+$$

$$+\sum_{j=2}^{i}\frac{d^j f}{dz^j}(\gamma(t))(P_{j-1,i}(\gamma^{'}(t),\gamma^{''}(t),...,\gamma^{(i)}(t))\gamma^{'}(t)+$$

$$\sum_{s=1}^{i}\frac{\partial P_{j,i}}{\partial z_s}(\gamma^{'}(t),\gamma^{''}(t),...,\gamma^{(i)}(t))\gamma^{(s+1)}(t)))+$$

$$+\frac{d^{i+1} f}{dz^{i+1}}(\gamma(t))(P_{i,i}(\gamma^{'}(t),\gamma^{''}(t),...,\gamma^{(i)}(t))\gamma^{'}(t)).$$

Thus, $g \in C^{i+1}(X)$,

$$P_{1,i+1}(z_1, z_2, ..., z_{i+1}) = \sum_{s=1}^{i}\frac{\partial P_{1,i}}{\partial z_s}(z_1, z_2, ..., z_i)z_{s+1},$$

$$P_{j,i+1}(z_1, z_2, ..., z_{i+1}) = \sum_{s=1}^{i}\frac{\partial P_{j,i}}{\partial z_s}(z_1, z_2, ..., z_i)z_{s+1}+$$
$$P_{j-1,i}(z_1, z_2, ..., z_i)z_1,$$

for $j = 2, 3, ..., i$, and

$$P_{i+1,i+1}(z_1, z_2, ..., z_{i+1}) = P_{i,i}(z_1, z_2, ..., z_i)z_1.$$

Therefore, the result holds also for $i + 1$ and the proof is complete. $\square$

Proposition 2.16. *Let $\gamma : I \to \mathbb{C}$ be a homeomorphism with $\gamma'(t) \neq 0$, where I is an interval or the unit circle and $k \in \{1, 2, 3, \cdots\} \cup \{\infty\}$. If $\gamma \in C^k(I)$, then $C^k(I) = C^k(\gamma^*) \circ \gamma$.*

Proof. By Lemma 2.15 we have that $C^k(I) \supset C^k(\gamma^*) \circ \gamma$. We will now prove by induction that $C^k(I) \subset C^k(\gamma^*) \circ \gamma$.
For $k = 1$, if $f \in C^1(I)$, we denote $g = f \circ \gamma^{-1}$. Then $g \circ \gamma = f$ and

$$\frac{dg}{dz}(\gamma(t_0)) = \lim_{t\to t_0}\frac{g(\gamma(t)) - g(\gamma(t_0))}{\gamma(t) - \gamma(t_0)}$$

$$= \lim_{t \to t_0} \frac{f(t) - f(t_0)}{t - t_0} \frac{1}{\frac{\gamma(t) - \gamma(t_0)}{t - t_0}} = \frac{f'(t_0)}{\gamma'(t_0)}$$

for $t_0 \in I$. So, the function $\frac{dg}{dz}$ exists and is continuous and thus $g \in C^1(\gamma^*)$. If the result is true for k, we will prove that it is also true for $k+1$. If $f \in C^{k+1}(I)$, we denote $g = f \circ \gamma^{-1}$. Then $g \circ \gamma = f$ and

$$\frac{dg}{dz}(\gamma(t_0)) = \frac{(f' \circ \gamma^{-1})(\gamma(t_0))}{\gamma'(t_0)}$$

for $t_0 \in I$. By the hypothesis of the induction, we have that $f' \circ \gamma^{-1} \in C^k(\gamma^*)$. It is also true that $\gamma' \in C^k(I)$. It follows that $\frac{dg}{dz} \in C^k(\gamma^*)$ or equivalently $g \in C^{k+1}(\gamma^*)$ and the proof is complete for k finite.
The case $k = \infty$ follows from the previous result for all k finite. $\square$

As we have showed above, if γ is a homeomorphism defined on X, then $C(X) = C(\gamma^*) \circ \gamma$. If also $\gamma \in C^k(X)$, where $k \in \{0, 1, 2, ...\} \cup \{\infty\}$, and $\gamma^{'}(t) \neq 0$, for every $t \in X$, then $C^k(X) = C^k(\gamma^*) \circ \gamma$. This implies that the spaces $C^k(\gamma^*)$ and $C^k(\gamma)$ contain exactly the same elements. Now, we will prove that they also share the same topology.

Proposition 2.17. *Let X be an interval $I \subset \mathbb{R}$ or the unit circle T, $k \in \{0, 1, 2, ...\} \cup \{\infty\}$ and γ be a homeomorphism defined on X, $\gamma \in C^k(X)$ and $\gamma^{'}(t) \neq 0$, for every $t \in X$. Then, the spaces $C^k(\gamma)$ and $C^k(\gamma^*)$ share the same topology.*

Proof. At first, we will prove the proposition in the special case where X is a compact interval $I \subset \mathbb{R}$ or the unit circle T and $k \neq \infty$. In order to do so, we will find $0 < M, N < \infty$, such that $d_1(f_1, f_2) \leq M d_2(f_1, f_2)$ and $d_2(f_1, f_2) \leq N d_1(f_1, f_2)$, for every $f_1, f_2 \in C^k(\gamma) = C^k(\gamma^*)$, where d_1, d_2 are the metrics of $C^k(\gamma)$ and $C^k(\gamma^*)$, respectively.

Let $f_1, f_2 \in C^k(\gamma^*)$ and $g_1 = f_1 \circ \gamma, g_2 = f_2 \circ \gamma \in C^k(X)$. We notice that

$$\sup_{t \in X} |g_1(t) - g_2(t)| = \sup_{z \in \gamma^*} |f_1(z) - f_2(z)| \leq d_2(f_1, f_2).$$

In addition,

$$(g_1 - g_2)^{(i)}(t) = \sum_{j=1}^{i} \frac{d^j(f_1 - f_2)}{dz^j}(\gamma(t)) P_{j,i}(\gamma^{'}(t), \gamma^{''}(t), ..., \gamma^{(i)}(t)), \quad (1)$$

for $1 \leq j \leq i \leq k$, where $P_{j,i}$ are the polynomials of Lemma 2.15. If $m_{j,i} = \sup_{t \in X} |P_{j,i}((\gamma^{'}(t), \gamma^{''}(t), ..., \gamma^{(i)}(t))|$ and $M_i = \max\{m_{j,i}, 1 \leq j \leq i\}$, then

$$\sup_{t \in X} |(g_1 - g_2)^{(i)}(t)| \leq \sum_{j=1}^{i} \sup_{z \in \gamma^*} |\frac{d^j(f_1 - f_2)}{dz^j}(z)| m_{j,i}$$

$$\leq M_i \sum_{j=1}^{i} \sup_{z \in \gamma^*} |\frac{d^j(f_1 - f_2)}{dz^j}(z)| \leq M_i d_2(f_1, f_2).$$

Consequently,

$$d_1(f_1, f_2) \leq M d_2(f_1, f_2),$$

where $M = 1 + M_1 + M_2 + ... + M_k$.

We also notice that

$$\sup_{z \in \gamma^*} |f_1(z) - f_2(z)| = \sup_{t \in X} |g_1(t) - g_2(t)| \leq d_1(f_1, f_2).$$

We will prove by induction that

$$\sup_{z \in \gamma^*} \left| \frac{d^i(f_1 - f_2)}{dz^i}(z) \right| \leq N_i d_1(f_1, f_2),$$

for some $N_i > 0$, $i = 1, 2, ...$, $i \leq k$. Since

$$P_{1,1}(z_1) = z_1$$

and

$$P_{i+1,i+1}(z_1, z_2, ..., z_{i+1}) = P_{i,i}(z_1, z_2, ..., z_i) z_1,$$

it is easy to see that $P_{i,i}(z_1, z_2, ..., z_i) = {z_1}^i$. Thus,

$$s_i = \inf_{t \in X} |P_{i,i}((\gamma^{'}(t), \gamma^{''}(t), ..., \gamma^{(i)}(t))| > 0,$$

because also $\gamma'(t) \neq 0$, for every $t \in X$. For $i = 1 \leq k$,

$$\sup_{z \in \gamma^*} \left| \frac{d(f_1 - f_2)}{dz}(z) \right| \leq \frac{1}{s_1} \sup_{t \in X} |(g_1 - g_2)'(t)| \leq \frac{1}{s_1} d_1(f_1, f_2),$$

and thus $N_1 = \dfrac{1}{s_1}$. If the result holds for every $1 \leq j \leq i < k$, then we will prove that it also holds for $i+1 \leq k$. Using our induction hypothesis and equation (1) we find

$$\sup_{t\in X} |(g_1 - g_2)^{(i+1)}(t)| \geq \sup_{z\in\gamma^*} \left| \frac{d^{i+1}(f_1 - f_2)}{dz^{i+1}}(z) \right| s_{i+1}$$

$$- \sum_{j=2}^{i} \sup_{z\in\gamma^*} \left| \frac{d^j(f_1 - f_2)}{dz^j}(z) \right| m_{j,i+1} \geq \sup_{z\in\gamma^*} \left| \frac{d^{i+1}(f_1 - f_2)}{dz^{i+1}}(z) \right| s_{i+1}$$

$$- \sum_{j=2}^{i} N_j m_{j,i+1} d_1(f_1, f_2).$$

Therefore,

$$\sup_{z\in\gamma^*} \left| \frac{d^{i+1}(f_1 - f_2)}{dz^{i+1}}(z) \right| \leq N_{i+1} d_1(f_1, f_2),$$

where $N_{i+1} = \dfrac{1}{s_{i+1}} + \sum_{j=2}^{i} \dfrac{N_j m_{j,i+1}}{s_{i+1}}$ and the result holds also for $i+1$. The induction is complete. Now, it is easy to see that

$$d_2(f_1, f_2) \leq N d_1(f_1, f_2),$$

where $N = 1 + N_1 + N_2 + ... + N_k$.

It easily follows from the above that even in the case where X is any type of interval $I \subset \mathbb{R}$ and/or $k = \infty$ the respective topologies of the spaces $C^k(\gamma)$ and $C^k(\gamma^*)$ are the same. The basic open subsets of $C^k(\gamma)$ are defined by a compact subset of X, an $l \in \{0, 1, 2, ...\}$, $l \leq k$, a function $f \in C^k(\gamma)$ and an $\varepsilon > 0$. But if we recall the definition of the topology of $C^k(\gamma^*)$ and use the above, we realize that this basic open subset of $C^k(\gamma)$ is also an open subset of $C^k(\gamma^*)$. Similarly, every basic open subset of $C^k(\gamma^*)$ is an open open subset of $C^k(\gamma)$. The proof is complete. □

Combining Propositions 2.14, 2.16 and 2.17, we obtain the following theorem.

Theorem 2.18. *Let $\gamma : I \to \mathbb{C}$ be a homeomorphism with $\gamma'(t) \neq 0$ for all $t \in I$, where I is an interval or the unit circle and $k \in \{1, 2, 3, \cdots\} \cup \{\infty\}$. Then $C^k(I) = C^k(\gamma^*) \circ \gamma$ if and only if $\gamma \in C^k(I)$. In addition the spaces $C^k(\gamma)$ and $C^k(\gamma^*)$ share the same topology.*

Remark 2.19. One can prove a slightly stronger statement than the one in Theorem 2.18. We do not need to assume $\gamma'(t) \neq 0$. Then $C^k(I) = C^k(\gamma^*) \circ \gamma$ if and only if $\gamma \in C^k(I)$ and $\gamma'(t) \neq 0$ for all $t \in I$.

Remark 2.20. With the definition of the derivative as in the Remark 2.12 we can define the spaces $C^k(E)$ for more general sets $E \subset \mathbb{C}$ but it may occur that the space $C^k(E)$ is not complete.

Theorem 2.18 shows that if γ is a homeomorphism and $\gamma'(t) \neq 0$ for $t \in I$, then $C^k(\gamma) \approx C^k(\gamma^*)$. Therefore, we have the the following corollary:

Corollary 2.21. *Let $\gamma : I \to \mathbb{C}$ be a homeomorphism with $\gamma'(t) \neq 0$ for $t \in I$, where I is an interval or the unit circle and $k \in \{1, 2, 3, \cdots\} \cup \{\infty\}$. Then, the space $C^k(\gamma)$ is independent of the parametrization of γ and coincides with the space $C^k(\gamma^*)$.*

3 Continuous analytic capacities

Next we present a few facts for the notion of continuous analytic capacity ([5]) that we will need in sections 5 and 7 below. Section 7 leads us to generalise this notion and thus consider the p-continuous analytic capacity.

Definition 3.1. *Let $U \subset \mathbb{C}$ be open. A function f belongs to the class $A(U)$ if $f \in H(U)$ and f has a continuous extension on $\overline{U}$, where the closure of U is taken in $\mathbb{C}$.*

Definition 3.2. *Let Ω be the complement in $\mathbb{C}$ of a compact set. A function f belongs to the class $A(\Omega \cup \{\infty\})$ if $f \in H(\Omega) \cap C(\Omega \cup \{\infty\})$ and f has a continuous extension on $\overline{\Omega}$, where the closure of Ω is taken in $\mathbb{C}$.*

By Tietze's extension theorem the extensions in both previous definitions can be considered as extensions on the whole of $\mathbb{C} \cup \{\infty\}$ without increase of the original norm $\|f\|_\infty$.

Definition 3.3. *Let L be a compact subset of $\mathbb{C}$. Let also $\Omega = \mathbb{C} \setminus L$. We denote*

$$a(L) = \sup\{|\lim_{z\to\infty} z(f(z) - f(\infty))| : f \in A(\Omega \cup \{\infty\}), \|f\|_\infty \leq 1\}$$

the continuous analytic capacity of L.

It is well known ([5]) that $a(L) = 0$ if and only if $A(\Omega \cup \{\infty\})$ contains only constant functions.

Theorem 3.4. *Let L be a compact subset of $\mathbb{C}$ and $U \subset \mathbb{C}$ be open with $L \subset U$. Then $a(L) = 0$ if and only if every $f \in C(U) \cap H(U \setminus L)$ belongs to $H(U)$.*

Proof. Assume that every $f \in C(U) \cap H(U \setminus L)$ belongs to $H(U)$.
We consider an arbitrary $f \in A(\Omega \cup \{\infty\})$. Since f can be continuously extended over L, it belongs to $C(U) \cap H(U \setminus L)$ and thus to $H(U)$. Therefore f is analytic in $\mathbb{C}$ and continuous in $\mathbb{C} \cup \{\infty\}$ and hence it is constant.
Thus $a(L) = 0$.
Now assume $a(L) = 0$ and we consider any $f \in C(U) \cap H(U \setminus L)$.
There exist two closed curves γ_1 and γ_2 in U so that γ_1 surrounds L and γ_2 surrounds γ_1. We define the analytic functions

$$\phi_1(z) = \frac{1}{2\pi i} \int_{\gamma_1} \frac{f(\zeta)}{\zeta - z}\, d\zeta \qquad \text{for } z \text{ in the exterior of } \gamma_1$$

and

$$\phi_2(z) = \frac{1}{2\pi i} \int_{\gamma_2} \frac{f(\zeta)}{\zeta - z}\, d\zeta \qquad \text{for } z \text{ in the interior of } \gamma_2.$$

Then the function g which equals $\phi_2 - f$ in the interior of γ_2 and ϕ_1 in the exterior of γ_1 is well defined and belongs to $A(\Omega \cup \{\infty\})$. Therefore g is constant and thus f is analytic in the interior of γ_2. Hence $f \in H(U)$. $\square$

Due to the local nature of the proof of the next theorem we shall state a few facts about the so-called Vitushkin's localization operator ([4]).

Let $U \subset \mathbb{C}$ be open and $f \in C(\mathbb{C}) \cap H(U)$. Let also $g \in C^1(\mathbb{C})$ have compact support. We define the function

$$\begin{aligned} G(z) &= \frac{1}{\pi} \iint \frac{f(z) - f(w)}{z - w} \frac{\partial g}{\partial \overline{w}}(w)\, dm(w) \\ &= f(z)g(z) - \frac{1}{\pi} \iint \frac{f(w)}{z - w} \frac{\partial g}{\partial \overline{w}}(w)\, dm(w). \end{aligned} \tag{2}$$

The function G is continuous in $\mathbb{C} \cup \{\infty\}$ with $G(\infty) = 0$, analytic in $U \cup (\mathbb{C} \setminus \operatorname{supp} g)$ and $f - G$ is analytic in the interior of the set $g^{-1}(\{1\})$.

Definition 3.5. *Let L be a closed subset of $\mathbb{C}$. We define*

$$a(L) = \sup\{a(M) : M \text{ compact subset of } L\}$$

the continuous analytic capacity of L.

Theorem 3.6. *Let L be a closed subset of $\mathbb{C}$. Then $a(L) = 0$ if and only if for every open set $U \subset \mathbb{C}$ every $f \in C(U) \cap H(U \setminus L)$ belongs to $H(U)$.*

Proof. One direction is immediate from Theorem 3.4 and Definition 3.5 and hence we assume that $a(L) = 0$.
We consider an arbitrary open set $U \subset \mathbb{C}$ which intersects L and an arbitrary $f \in C(U) \cap H(U \setminus L)$ and we shall prove that f extends analytically over $U \cap L$.
Now L may not be contained in U but since analyticity is a local property we shall employ Vitushkin's localization operator.
Let $z_0 \in U \cap L$ and $\overline{D(z_0, 3\delta)} \subset U$. We consider $g \in C^1(\mathbb{C})$ with $\operatorname{supp} g \subset \overline{D(z_0, 2\delta)}$ such that $g = 1$ in $D(z_0, \delta)$.
We also consider the restriction F of f in $\overline{D(z_0, 3\delta)}$ and we extend F so that it is continuous in $\mathbb{C} \cup \{\infty\}$.
We define as in (2) the function

$$G(z) = \frac{1}{\pi} \iint \frac{F(z) - F(w)}{z - w} \frac{\partial g}{\partial \overline{w}}(w)\, dm(w).$$

Now G is continuous in $\mathbb{C} \cup \{\infty\}$ with $G(\infty) = 0$, analytic in $(D(z_0, 3\delta) \setminus L) \cup (\mathbb{C} \setminus D(z_0, 2\delta)) = \mathbb{C} \setminus (\overline{D(z_0, 2\delta)} \cap L)$ and $f - G = F - G$ is analytic in $D(z_0, \delta)$.
Since $a(L) = 0$, we have $a(\overline{D(z_0, 2\delta)} \cap L) = 0$ and hence G is constant 0 in $\mathbb{C}$. Therefore f is analytic in $D(z_0, \delta)$.
Since $z_0 \in U \cap L$ is arbitrary we conclude that $f \in H(U)$. □

Theorem 3.7. *([5]) If L is a Jordan arc with locally finite length, then $a(L) = 0$. The same holds for any countable union of such curves. Therefore, line segments, circular arcs, analytic curves and boundaries of convex sets are all of zero continuous analytic capacity.*

Definition 3.8. *Let U be an open subset of $\mathbb{C}$ and $p \in \{0, 1, 2, \cdots\} \cup \{\infty\}$. A function f belongs to the class $A^p(U)$ if $f \in H(U)$ and all derivatives $f^{(j)}$, $j \in \{0, 1, 2, \cdots\}$, $0 \leq j \leq p$, have continuous extensions $f^{(j)}$ on $\overline{U}$, where the closure of U is taken in $\mathbb{C}$.*

Definition 3.9. *Let Ω be the complement in $\mathbb{C}$ of a compact set and $p \in \{0, 1, 2, \cdots\} \cup \{\infty\}$. A function f belongs to the class $A^p(\Omega \cup \{\infty\})$ if $f \in H(\Omega) \cap C(\Omega \cup \{\infty\})$ and all derivatives $f^{(j)}$, $j \in \{0, 1, 2, \cdots\}$, $0 \leq j \leq p$, have continuous extensions $f^{(j)}$ on $\overline{\Omega}$, where the closure of Ω is taken in $\mathbb{C}$.*

Definition 3.10. *Let L be a compact subset of $\mathbb{C}$ and $p \in \{0, 1, 2, \cdots\} \cup \{\infty\}$. Let also $\Omega = \mathbb{C} \setminus L$. For $p \neq \infty$, we denote the p-continuous analytic capacity as*

$$a_p(L) = \sup\{|\lim_{z\to\infty} z(f(z) - f(\infty))| : f \in A^p(\Omega \cup \{\infty\}), \max_{0\leq j\leq p} \|f^{(j)}\|_\infty \leq 1\}.$$

Obviously, $a_0(L) = a(L)$.
For $p = \infty$,

$$a_\infty(L) = \sup\{|\lim_{z\to\infty} z(f(z) - f(\infty))| : f \in A^\infty(\Omega \cup \{\infty\}), d(f, 0) \leq 1\},$$

where the Frechet distance $d(f, 0)$ is defined by

$$d(f, 0) = \sum_{j=0}^{\infty} 2^{-j} \frac{\|f^{(j)}\|_\infty}{1 + \|f^{(j)}\|_\infty}$$

If L is a closed subset of $\mathbb{C}$ then we define

$$a_p(L) = \sup\{a_p(M) : M \text{ compact subset of } L\}.$$

Definition 3.11. *Let L be a compact subset of $\mathbb{C}$ and $p \in \{0, 1, 2, \cdots\} \cup \{\infty\}$. Let also $\Omega = \mathbb{C} \setminus L$. We denote*

$$a'_p(L) = \sup\{|\lim_{z\to\infty} z(f(z) - f(\infty))| : f \in A^p(\Omega \cup \{\infty\}), \|f\|_\infty \leq 1\}.$$

Obviously, $a'_0(L) = a(L)$.
If L is a closed subset of $\mathbb{C}$ then we define

$$a'_p(L) = \sup\{a'_p(M) : M \text{ compact subset of } L\}.$$

It is obvious that $a_p(L)$ and $a'_p(L)$ are decreasing function of p.

The following theorem corresponds to Theorem 3.4.

Theorem 3.12. *Let L be a compact subset of $\mathbb{C}$ and $U \subset \mathbb{C}$ be open with $L \subset U$ and let $p \in \{0, 1, 2, \cdots\} \cup \{\infty\}$. Then $a_p(L) = 0$ if and only if every $f \in A^p(U \setminus L)$ has an extension to $A^p(U)$, if and only if $a'_p(L) = 0$.*

The proof is a repetition of the proof of Theorem 3.4.

Definition 3.13. *Let L be a compact subset of $\mathbb{C}$. For $p \in \{0, 1, 2, \cdots\} \cup \{\infty\}$ we define $b_p(L)$ such that $b_p(L) = 0$ when $a_p(L) = 0$ and $b_p(L) = \infty$ when $a_p(L) \neq 0$. For $p, q \in \{0, 1, 2, \cdots\} \cup \{\infty\}$ we will say that $a_p(L)$ and $a_q(L)$ are essentially different if $b_p(L) \neq b_q(L)$.*

Definition 3.14. *Let U be an open subset of $\mathbb{C}$ and $p \in \{1, 2, \cdots\} \cup \{\infty\}$. A function f belongs to the class $\tilde{A}^p(U)$ if $f \in A^p(U)$ and for $0 \leq j \leq j' \leq p$ the following is true for $z, w \in \overline{U}$:*

$$f^{(j)}(w) - \sum_{k=0}^{j'-j} \frac{1}{k!} f^{(j+k)}(z)(w - z)^k = o(|w - z|^{j'-j}) \qquad \text{as } w \to z. \tag{3}$$

This is supposed to hold uniformly for z, w in compact subsets of $\overline{U}$. The definition of the space $\tilde{A}^p(\Omega \cup \{\infty\})$ is analogous to the Definition 3.9 of $A^p(\Omega \cup \{\infty\})$.

Note that, since $f \in H(U)$, relation (3) is automatically true for $z \in U$ and thus the "point" of the definition is when $z \in \partial U$.

If p is finite, then $f \in \tilde{A}^p(U)$ admits as a norm $\|f\|_{\tilde{A}^p(U)}$ the smallest M such that

$$|f^{(j)}(z)| \leq M \qquad \text{for } z \in \overline{U}, 0 \leq j \leq p,$$

$$\left| f^{(j)}(w) - \sum_{k=0}^{j'-j} \frac{1}{k!} f^{(j+k)}(z)(w - z)^k \right| \leq M|w - z|^{j'-j} \quad w, z \in \overline{U}, |w - z| \leq 1,$$

$$0 \leq j \leq j' \leq p.$$

It is easy to see that $\tilde{A}^p(U)$ with this norm is complete.

If p is infinite, then, using the norms for the finite cases in the standard way, $\tilde{A}^\infty(U)$ becomes a Frechet space.

There is a fundamental result of Whitney ([13]) saying that if $f \in \tilde{A}^p(U)$, then f can be extended in $\mathbb{C}$ in such a way that the extended f belongs to $C^p(\mathbb{C})$ and that the partial derivatives of f of order $\leq p$ in $\mathbb{C}$ are extensions of the original partial derivatives of f in U.

Definition 3.15. *Let L be a compact subset of $\mathbb{C}$ and $p \in \{0,1,2,\cdots\} \cup \{\infty\}$. Let also $\Omega = \mathbb{C} \setminus L$. For $p \neq \infty$, we denote*

$$\tilde{a}_p(L) = \sup\{|\lim_{z\to\infty} z(f(z) - f(\infty))| : f \in \tilde{A}^p(\Omega \cup \{\infty\}), \|f\|_{\tilde{A}^p(U)} \leq 1\}.$$

In the case $p = \infty$ the norm $\|f\|_{\tilde{A}^p(U)}$ is replaced by the distance of f from 0 in the metric space structure of $\tilde{A}^\infty(U)$.
Obviously, $\tilde{a}_0(L) = a(L)$.
If L is a closed subset of $\mathbb{C}$ then we define

$$\tilde{a}_p(L) = \sup\{\tilde{a}_p(M) : M \text{ compact subset of } L\}.$$

It turns out that, in the case $p \geq 1$, there is a simple topological characterization of the compact sets L with $\tilde{a}_p(L) = 0$.

Theorem 3.16. *Let L be a compact subset of $\mathbb{C}$ and $p \geq 1$. Then $\tilde{a}_p(L) = 0$ if and only if L has empty interior.*

Proof. Let L have nonempty interior and let the disc $D(z_0, r_0)$ be contained in L. Obviously, the non-constant function $\frac{1}{z-z_0}$ belongs to $\tilde{A}^p(\Omega \cup \{\infty\})$ for all p and thus $a_p(L) > 0$ for all p.

Conversely, let L have empty interior and let f belong to $\tilde{A}^1(\Omega \cup \{\infty\})$. Then f is analytic in Ω and at every $z \in L$ we have

$$f(w) - f(z) - f'(z)(w-z) = o(|w-z|) \qquad \text{as } w \to z, w \in \overline{\Omega} = \mathbb{C}.$$

Thus f is analytic at z (with derivative equal to $f'(z)$) and hence analytic in all of $\mathbb{C}$. Since f is continuous at ∞, it is a constant. Therefore $\tilde{A}^1(\Omega \cup \{\infty\})$ contains only the constant functions and $\tilde{a}_1(L) = 0$. $\square$

Theorem 3.17. *There is a compact subset L of $\mathbb{C}$ such that $\tilde{a}_1(L) = 0 < \tilde{a}_0(L)$.*

Proof. Due to the last theorem, it is enough to find a compact L with empty interior and with $\tilde{a}_0(L) = a(L) > 0$.

This set L is a Cantor type set. We consider a sequence (a_n) with $0 < a_n < \frac{1}{2}$ for all $n = 1, 2, 3, \ldots$ and construct a sequence (L_n) of decreasing compact sets as follows. L_0 is the unit square $[0,1] \times [0,1]$ and L_1 is the union of the four squares at the four corners of L_0 with sidelength equal to

a_1. We then continue inductively. If L_n is the union of 4^n squares each of sidelength equal to

$$l_n = a_1 \cdots a_n,$$

then each of these squares produces four squares at its four corners each of sidelength equal to $a_1 \cdots a_n a_{n+1}$. The union of these new squares is L_{n+1}.

We denote $I_{n,k}$, $k = 1, \ldots, 4^n$, the squares whose union is L_n.

Finally, we define

$$L = \bigcap_{n=1}^{+\infty} L_n.$$

It is clear that L is a totally disconnected compact set. The area of L_n equals

$$|L_n| = 4^n (a_1 \cdots a_n)^2 = (2a_1 \cdots 2a_n)^2.$$

Now we assume that

$$\sum_{n=1}^{+\infty} (1 - 2a_n) < +\infty.$$

Under this condition we find that the area of L equals

$$|L| = \lim_{n \to +\infty} |L_n| = \lim_{n \to +\infty} (2a_1 \cdots 2a_n)^2 > 0.$$

Then it is well known ([5]) that the function

$$f(z) = \frac{1}{\pi} \iint_L \frac{1}{z - w} \, dm(w)$$

is continuous in $\mathbb{C} \cup \{\infty\}$ with $f(\infty) = 0$ and holomorphic in $\mathbb{C} \setminus L$. Since

$$\lim_{z \to \infty} z f(z) = |L| > 0,$$

f is not identically equal to 0 and hence $\tilde{a}_0(L) = a(L) > 0$. $\square$

Remark 3.18. The latter part of the above proof shows that if a compact set L (not necessarily of Cantor type) has strictly positive area, then $a(L) > 0$, which is a well known fact ([5]).

The problem of the characterization of the compact sets L with $a_p(L) = 0$ seems to be more complicated.

We will show that there exists a compact set L such that $a_0(L)$ and $a_1(L)$ are essentially different; that is $a_0(L) > 0$ and $a_1(L) = 0$.

Theorem 3.19. *There is a compact subset L of $\mathbb{C}$ such that $a_1(L) = 0 < a_0(L)$.*

Proof. We consider the same Cantor type set L which appeared in the proof of the previous theorem. We keep the same notation.

We now take any f which belongs to $A^1(\Omega \cup \{\infty\})$. Subtracting $f(\infty)$ from f, we may also assume that $f(\infty) = 0$.

Let $z_0 \in \Omega$. Then there is n_0 such that $z_0 \notin L_n$ for all $n \geq n_0$.

By Cauchy's formula, for every $n \geq n_0$ we have

$$f(z_0) = -\frac{1}{2\pi i} \sum_{k=1}^{4^n} \int_{\gamma_{n,k}} \frac{f(z)}{z - z_0}\, dz \tag{4}$$

where $\gamma_{n,k}$ is the boundary curve of the square $I_{n,k}$.

Let $z_{n,k}$ be any point of Ω inside $I_{n,k}$ (for example, the center of the square). It is geometrically obvious that for every $z \in \gamma_{n,k}$ there is a path (consisting of at most two line segments) γ with length $l(\gamma) \leq 2l_n$ joining z and $z_{n,k}$ and contained in Ω (with the only exception of its endpoint z). Since $f \in A^1(\Omega \cup \{\infty\})$, we get

$$\begin{aligned} |f(z) - f(z_{n,k}) - f'(z_{n,k})(z - z_{n,k})| &= \Big| \int_\gamma (f'(\zeta) - f'(z_{n,k}))\, d\zeta \Big| \\ &\leq \int_\gamma |f'(\zeta) - f'(z_{n,k})|\, d\zeta \leq \epsilon_n l(\gamma) \leq 2\epsilon_n l_n, \end{aligned}$$

where $\epsilon_n \to 0$ uniformly for $z \in \gamma_{n,k}$ and for $k = 1, \ldots, 4^n$.

Therefore

$$\Big| \int_{\gamma_{n,k}} \frac{f(z)}{z - z_0}\, dz \Big| = \Big| \int_{\gamma_{n,k}} \frac{f(z) - f(z_{n,k}) - f'(z_{n,k})(z - z_{n,k})}{z - z_0}\, dz \Big| \leq \epsilon_n \frac{8l_n^2}{\delta_0}$$

where δ_0 is the distance of z_0 from L_{n_0}.

Thus from (4) we get

$$|f(z_0)| \leq \frac{8 \cdot 4^n l_n^2}{\pi \delta_0}\, \epsilon_n = \frac{8|L_n|}{\pi \delta_0}\, \epsilon_n \leq \frac{8}{\pi \delta_0}\, \epsilon_n.$$

This holds for all $n \geq n_0$ and hence $f(z_0) = 0$ for all $z_0 \in \Omega$.

We proved that the only element f of $A^1(\Omega \cup \{\infty\})$ with $f(\infty) = 0$ is the zero function and thus $a_1(L) = 0$. $\square$

We will now see a different proof of the above theorem. The proof is longer from the previous one, but it provides us a more general result.

Theorem 3.20. *Let $K_{1/3}$ be the usual Cantor set lying on $[0,1]$ and $L = K_{1/3} \times K_{1/3}$. Then $a_0(L) > 0$, but $a_1(L) = 0$.*

Proof. First, we observe that the area of L is 0, as

$$L = \bigcap_{n=0}^{\infty} L_n,$$

where each L_n is the union of 4^n squares of area 9^{-n}. It is known ([6], [14]) that there exists a function g continuous on S^2 and holomorphic off L, such that

$$g'(\infty) = \lim_{z\to\infty} z(g(z) - g(\infty)) \neq 0,$$

which implies that $a_0(L) > 0$.

We will prove that $a_1(L) = 0$ or equivalently that every function in $A^1(\Omega \cup \{\infty\})$ is entire, where $\Omega = \mathbb{C} \setminus L$. Let $f \in A^1(\Omega \cup \{\infty\})$, $\varepsilon > 0$ and $\varphi_\varepsilon = \varepsilon^{-2}\chi_\varepsilon$, where χ_ε is the characteristic function of the square S_ε with center at 0 and sides parallel to the axes with length ε. It is easy to see from the continuity of f that the convolutions $f * \varphi_\varepsilon$ belong to $C^1(\mathbb{C})$ and converge uniformly on $D(0,2)$ to f as $\varepsilon \to 0$. Since $f \in A^1(\Omega \cup \{\infty\})$, the partial derivatives $\dfrac{\partial u}{\partial x}$, $\dfrac{\partial u}{\partial y}$ and $\dfrac{\partial v}{\partial x}$, $\dfrac{\partial v}{\partial y}$ of $u = Ref$ and $v = Imf$, respectively, extend continuously on $\mathbb{C} \cup \{\infty\}$ and hence are bounded. We will prove that

$$\frac{\partial u * \varphi_\varepsilon}{\partial x} = \frac{\partial u}{\partial x} * \varphi_\varepsilon \tag{5}$$

on Ω. Let $(a,b) \in \Omega$ and $h \in \mathbb{R}$. Then

$$\frac{(u * \varphi_\varepsilon)(a+h,b) - (u * \varphi_\varepsilon)(a,b)}{h} =$$

$$\varepsilon^{-2} \iint_{S_\varepsilon} \frac{u(a+h-x, b-y) - u(a-x, b-y)}{h} dxdy.$$

It is easy to see that for almost every $(x,y) \in S_\varepsilon$ and every $h \in \mathbb{R}$ the segment $[(a+h-x, b-y), (a-x, b-y)]$ is a subset of Ω. Thus, from Mean value theorem for almost every $(x, y) \in S_\varepsilon$ and every $h \in \mathbb{R}$ there exists $q \in \mathbb{R}$, such that

$$\frac{u(a+h-x, b-y) - u(a-x, b-y)}{h} = \frac{\partial u}{\partial x}(a+q-x, b-y),$$

which remains bounded by a constant $M > 0$. For those $(x, y) \in S_\varepsilon$

$$\frac{u(a+h-x, b-y) - u(a-x, b-y)}{h}$$

converges to

$$\frac{\partial u}{\partial x}(a-x, b-y),$$

as h converges to 0. By Dominated convergence theorem (5) holds true. Similarly,

$$\frac{\partial u * \varphi_\varepsilon}{\partial y} = \frac{\partial u}{\partial y} * \varphi_\varepsilon, \frac{\partial v * \varphi_\varepsilon}{\partial x} = \frac{\partial v}{\partial x} * \varphi_\varepsilon, \frac{\partial v * \varphi_\varepsilon}{\partial y} = \frac{\partial v}{\partial y} * \varphi_\varepsilon.$$

Since the Cauchy-Riemann equations are satisfied for f almost everywhere, we have that

$$\frac{\partial u}{\partial x} * \varphi_\varepsilon = \frac{\partial v}{\partial y} * \varphi_\varepsilon$$

and

$$\frac{\partial u}{\partial y} * \varphi_\varepsilon = -\frac{\partial v}{\partial x} * \varphi_\varepsilon.$$

Thus, the Cauchy-Riemann equations are satisfied for every $f * \varphi_\varepsilon$ on Ω. Since the interior of L is void, the set Ω is dense in $\mathbb{C}$. From the continuity of the partial derivatives of every $f * \varphi_\varepsilon$ on $\mathbb{C}$, the Cauchy-Riemann equations are satisfied for every $f * \varphi_\varepsilon$ on $\mathbb{C}$, which implies that every $f * \varphi_\varepsilon$ is holomorphic on $\mathbb{C}$. Finally, the functions $f * \varphi_\varepsilon$ converge uniformly on $D(0, 2)$ to f, as $\varepsilon \to 0$, which combined with Weierstrass theorem implies that f is holomorphic on $D(0, 2)$ and therefore f is holomorphic on $\mathbb{C}$. □

Remark 3.21. The above proof also shows that if L is a compact subset of $\mathbb{C}$ with zero area and if almost for every line ε which is parallel to the x-axis and almost for every line ε which is parallel to the y-axis, $\varepsilon \cap L = \emptyset$, then $a_1(L) = 0$. In fact, it suffices that these intersections are finite for a dense set of ε parallel to the x-axis and for a dense set of ε parallel to the y-axis.

4 Real analyticity on analytic curves

Let $L \subset \mathbb{C}$ be a closed set without isolated points. We denote by $C(L)$ the set of continuous functions $f : L \to \mathbb{C}$. This space endowed with the topology of uniform convergence on the compact subsets of L is a complete metric space and thus Baire's theorem is at our disposal.

Lemma 4.1. *Let $L \subset \mathbb{C}$ be a closed set without isolated points. Let also $z_0 \in L$ be the center and $r > 0$ be the radius of the open disk $D(z_0, r)$ and $0 < M < +\infty$. The set of continuous functions $f : L \to \mathbb{C}$, for which there exists a holomorphic and bounded by M on $D(z_0, r)$ function F such that $F|_{D(z_0,r)\cap L} = f|_{D(z_0,r)\cap L}$, is a closed subset of $C(L)$ and has empty interior.*

Proof. Let $A(M, z_0, r)$ be the set of continuous functions $f : L \to \mathbb{C}$ for which there exists a holomorphic and bounded by M on $D(z_0, r)$ function F, such that $F|_{D(z_0,r)\cap L} = f|_{D(z_0,r)\cap L}$; That is we assume that $|F(z)| \leq M$ for all $z \in D(z_0, r)$. We distinguish two cases according to whether $D(z_0, r)$ is contained or not in L.

1) If $D(z_0, r) \subset L$, then the elements of $A(M, z_0, r)$ belong to $C(L)$ and are holomorphic and bounded by M on $D(z_0, r)$. Let $(f_n)_{n\geq 1}$ be a sequence in $A(M, z_0, r)$ converging uniformly on the compact subsets of L to a function f defined on L. Then, from Weierstrass theorem, it follows that f will be holomorphic and bounded by M on $D(z_0, r)$. Therefore, $f \in A(M, z_0, r)$ and $A(M, z_0, r)$ is a closed subset of $C(L)$.

If $A(M, z_0, r)$ has not empty interior, then there exists a function f in the interior of $A(M, z_0, r)$, a compact set $K \subset L$ and $\delta > 0$ such that

$$\left\{ g \in C(L) : \sup_{z\in K} |f(z) - g(z)| < \delta \right\} \subset A(M, z_0, r).$$

Then the function $h(z) = f(z) + \frac{\delta}{2}\bar{z}$, $z \in L$ belongs to $A(M, z_0, r)$ and therefore is holomorphic on $D(z_0, r)$. But then the function $\frac{\delta}{2}\bar{z}$ will be holomorphic on $D(z_0, r)$, which is absurd. Thus, the interior of $A(M, z_0, r)$ is void.

2)If $D(z_0, r)$ is not contained in L, then there exists $w \in D(z_0, r)\setminus L$. Let $(f_n)_{n\geq 1}$ be a sequence in $A(M, z_0, r)$ where f_n converges uniformly on compact subsets of L to a function f defined on L. Then, for $n = 1, 2, \ldots$ there are holomorphic functions $F_n : D(t_0, r) \to \mathbb{C}$ bounded by M such that $F_n|_{D(z_0,r)\cap L} = f_n|_{D(z_0,r)\cap L}$. By Montel's theorem, there exists a subsequence

(F_{k_n}) of (F_n) which converges uniformly to a function F on the compact subsets of $D(z_0, r)$ which is holomorphic and bounded by M on $D(z_0, r)$. Because $F_{k_n} \to f$ at $D(z_0, r) \cap L$ we have that $F|_{D(z_0,r)\cap L} = f|_{D(z_0,r)\cap L}$ and so $f \in A(M, z_0, r)$. Therefore, $A(M, z_0, r)$ is a closed subset of $C(L)$.

If $A(M, z_0, r)$ has not empty interior, then there exists a function f in the interior of $A(M, z_0, r)$, a compact set $K \subset L$ and $\delta > 0$ such that

$$\left\{ g \in C(L) : \sup_{z \in K} |f(z) - g(z)| < \delta \right\} \subset A(M, z_0, r).$$

We choose $0 < a < \delta \inf_{z\in K} |z - w|$. We notice that this is possible because $\inf_{z\in K} |z - w| > 0$, since $w \notin L$ and $K \subset L$. The function $h(z) = f(z) + \dfrac{a}{2(z - w)}$ for $z \in L$ belongs to $A(M, z_0, r)$ and therefore it has a holomorphic and bounded extension H on $D(z_0, r)$, such that $H|_{D(z_0,r)\cap L} = h|_{D(z_0,r)\cap L}$. However, there exists a holomorphic function $F : D(z_0, r) \to \mathbb{C}$ which coincides with f on $D(z_0, r) \cap L$. By analytic continuation $H(z) = F(z) + \dfrac{a}{2(z - w)}$ for $z \in D(z_0, r) \setminus \{z_0\}$, since they are equal on $L \cap (D(z_0, r) \setminus \{z_0\})$, which contains infinitely many points close to z_0, all of them being non isolated. As a result H is not bounded at $D(z_0, r)$ which is a contradiction. Thus, $A(M, z_0, r)$ has empty interior. □

Definition 4.2. *Let $L \subset \mathbb{C}$ be a closed set without isolated points and $z_0 \in L$. A function $f \in C(L)$ belongs to the class of non-holomorphically extendable at z_0 functions defined and continuous on L if there exists no pair of an open disk $D(z_0, r)$, $r > 0$ and a holomorphic function $F : D(z_0, r) \to \mathbb{C}$, such that $F|_{D(z_0,r)\cap L} = f|_{D(z_0,r)\cap L}$.*

Theorem 4.3. *Let $L \subset \mathbb{C}$ be a closed set without isolated points and $z_0 \in L$. The class of non-holomorphically extendable at z_0 functions defined and continuous on L is a dense and G_δ subset of $C(L)$.*

Proof. The set

$$\bigcap_{n=1}^{\infty} \bigcap_{M=1}^{\infty} \left(C(L) \setminus A\left(M, z_0, \frac{1}{n} \right) \right)$$

is a dense G_δ subset of $C(L)$ according to Baire's Theorem and coincides with the class of non-holomorphically extendable at z_0 functions defined and continuous on L, because every holomorphic function on $D(z_0, r)$ becomes bounded if we restrict it on $D(z_0, r')$ for $r' < r$. □

Definition 4.4. *Let $L \subset \mathbb{C}$ be a closed set without isolated points. A function $f \in C(L)$ belongs to the class of nowhere holomorphically extendable functions defined and continuous on L, if there exists no pair of an open disk $D(z_0, r), z_0 \in L, r > 0$ and a holomorphic function $F : D(z_0, r) \to \mathbb{C}$, such that $F|_{D(z_0,r)\cap L} = f|_{D(z_0,r)\cap L}$.*

Theorem 4.5. *Let $L \subset \mathbb{C}$ be a closed set without isolated points. The class of nowhere holomorphically extendable functions defined and continuous on L is a dense and G_δ subset of $C(L)$.*

Proof. Let $z_l, l = 1, 2, 3, ...$ be a dense in L sequence of points of L. Then the set

$$\bigcap_{l=1}^{\infty}\bigcap_{n=1}^{\infty}\bigcap_{M=1}^{\infty}\left(C(L)\setminus A\left(M, z_l, \frac{1}{n}\right)\right)$$

is a dense G_δ subset of $C(L)$ according to Baire's Theorem. This set coincides with the class of nowhere holomorphically extendable functions defined and continuous on L, because every holomorphic function on $D(z_0, r)$ becomes bounded if we restrict it on $D(z_0, r')$ for $r' < r$. $\square$

The proof of the above results can be used to prove similar results at some special cases. Let $\gamma : I \to \mathbb{C}$ be a a continuous and locally injective curve, where I is an interval in $\mathbb{R}$ of any type. The symbol γ^* will be used instead of $\gamma(I)$. It is obvious that γ^* has no isolated points. We also recall Definition 2.2 of $C^k(\gamma)$.

Definition 4.6. *Let $\gamma : I \to \mathbb{C}$ be a locally injective curve and $z_0 = \gamma(t_0), t_0 \in I$, where I is an interval. A function $f : \gamma^* \to \mathbb{C}$ belongs to the class of non-holomorphically extendable at $(t_0, z_0 = \gamma(t_0))$ functions, if there are no open disk $D(z_0, r), r > 0$ and $\eta > 0$ and a holomorphic function $F : D(z_0, r) \to \mathbb{C}$, such that $\gamma((t_0 - \eta, t_0 + \eta) \cap I) \subset D(z_0, r)$ and $F(\gamma(t)) = f(\gamma(t))$ for all $t \in (t_0 - \eta, t_0 + \eta) \cap I$. Otherwise we say that f is holomorphically extendable at $(t_0, z_0 = \gamma(t_0))$.*

Theorem 4.7. *Let $k, l \in \{0, 1, 2, ...\} \cup \{\infty\}$ such that $l \leq k$. Let also $\gamma : I \to \mathbb{C}$ be a locally injective function belonging to $C^k(I)$, and $t_0 \in I$, where I is an interval. The class of non-holomorphically extendable at $(t_0, z_0 = \gamma(t_0))$ functions belonging to $C^l(\gamma)$ is a dense and G_δ subset of $C^l(\gamma)$.*

Proof. Let $r > 0$ and $\eta > 0$ such that $\gamma(t_0 - \eta, t_0 + \eta) \subset D(\gamma(t_0), r)$. Let also $A(M, z_0, r, \eta, l)$ be the set of $C^l(I)$ functions for which there exists a holomorphic and bounded by M on $D(z_0, r)$ function F, such that $F(\gamma(t)) = f(\gamma(t))$ for all $t \in (t_0 - \eta, t_0 + \eta) \cap I$; That is we assume that $|F(z)| \leq M$ for all $z \in D(z_0, r)$.

Since $\gamma \in C^k(I)$, the open disk $D(z_0, r)$ is not contained in γ^* (see Proposition 6.2) and thus there exists $w \in D(z_0, r) \setminus \gamma^*$. Similarly to Lemma 4.1, $A(M, z_0, r, \eta, l)$ is a closed subset of $C^l(\gamma)$.

If $A(M, z_0, r, \eta, l)$ has not empty interior, then there exists a function f in the interior of $A(M, z_0, r, \eta, l)$, $b \in \{0, 1, 2, ...\}$, a compact set $K \subset I$ and $\delta > 0$ such that

$$\{g \in C^k(\gamma) : \sup_{t \in K} \left|(f \circ \gamma)^{(j)}(t) - (g \circ \gamma)^{(j)}(t)\right| < \delta,$$
$$0 \leq j \leq b\} \subset A(M, z_0, r, \eta, l).$$

We choose $0 < a < \delta \min\{\inf_{t \in K} |\gamma(t) - w|, \inf_{t \in K} |\gamma(t) - w|^2, ..., \frac{1}{b!} \inf_{t \in K} |\gamma(t) - w|^{b+1}\}$. This is possible because $w \notin \gamma^*$ and $\gamma(K) \subset \gamma^*$. The function $h(z) = f(z) + \dfrac{a}{2(z - w)}$ for $z \in \gamma^*$ belongs to $A(M, z_0, r, \eta, l)$, since $\gamma \in C^k(I)$. Similarly to Lemma 4.1, we are lead to a contradiction. Therefore, $A(M, z_0, r, \eta, l)$ has empty interior.

Let $s_{n,m}$, $n = 1, 2, 3, ...$, $m = 1, 2, 3, ...$ be a sequence such that $\lim_{m \to \infty} s_{n,m} = 0$ for every $n = 1, 2, 3, ...$ and $\gamma(t_0 - s_{n,m}, t_0 + s_{n,m}) \subset D\left(\gamma(t_0), \frac{1}{n}\right)$ for every $n = 1, 2, 3, ...$ and every $m = 1, 2, 3, ...$ Then the class of non-holomorphically extendable at $(t_0, z_0 = \gamma(t_0))$ functions belonging to $C^l(\gamma)$ coincides with the set

$$\bigcap_{n=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcap_{M=1}^{\infty} \left(C^l(\gamma) \setminus A\left(M, z_0, \frac{1}{n}, s_{n,m}, l\right)\right),$$

because every holomorphic function on $D(z_0, r)$ becomes bounded if we restrict it on $D(z_0, r')$ for $r' < r$. Thus, according to Baire's theorem the class of non-holomorphically extendable at $(t_0, z_0 = \gamma(t_0))$ functions belonging to $C^l(\gamma)$ is a dense and G_δ subset of $C^l(\gamma)$. □

Definition 4.8. *Let $\gamma : I \to \mathbb{C}$ be a locally injective curve, where I is an interval. A function $f : \gamma^* \to \mathbb{C}$ belongs to the class of nowhere holomorphically extendable functions if there are no open disk $D(z_0, r), z_0 =$*

$\gamma(t_0), t_0 \in I, r > 0$ *and* $\eta > 0$ *and a holomorphic function* $F : D(z_0, r) \to \mathbb{C}$, *such that* $\gamma((t_0 - \eta, t_0 + \eta) \cap I) \subset D(z_0, r)$ *and* $F(\gamma(t)) = f(\gamma(t))$ *for all* $t \in (t_0 - \eta, t_0 + \eta) \cap I$.

Theorem 4.9. *Let* $k, l \in \{0, 1, 2, ...\} \cup \{\infty\}$ *such that* $l \leq k$. *Let also* $\gamma : I \to \mathbb{C}$ *be a locally injective function belonging to* $C^k(I)$, *where* I *is an interval. The class of nowhere holomorphically extendable functions belonging to* $C^l(\gamma)$ *is a dense and* G_δ *subset of* $C^l(\gamma)$.

Proof. Let t_n, $n = 1, 2, 3, ...$ be a dense in I sequence of points of I. Then the class of nowhere holomorphically extendable functions belonging to $C^k(\gamma)$ coincides with the intersection over every $n = 1, 2, 3, ...$ of the classes of non-holomorphically extendable at $(t_n, z_n = \gamma(t_n))$ functions belonging to $C^l(\gamma)$. Since the classes of non-holomorphically extendable at $(t_n, z_n = \gamma(t_n))$ functions belonging to $C^l(\gamma)$ are dense and G_δ subsets of $C^l(\gamma)$ according to Theorem 4.7, it follows that the class of nowhere holomorphically extendable functions belonging to $C^l(\gamma)$ is a dense and G_δ subset of $C^l(\gamma)$ from Baire's theorem. □

Now,using results of non-extendability, we will prove results for real analyticity.

Proposition 2.9 and Theorem 4.9 immediately prove the following theorems.

Theorem 4.10. *Let* $\gamma : I \to \mathbb{C}$ *be an analytic curve, where* I *is an interval and* $t_0 \in I$. *For* $k = 0, 1, 2, \ldots$ *or* $k = \infty$ *the class of functions* $f \in C^k(\gamma)$ *which are not real analytic at* $(t_0, z_0 = \gamma(t_0))$ *is a dense and* G_δ *subset of* $C^k(\gamma)$.

Theorem 4.11. *Let* $\gamma : I \to \mathbb{C}$ *be an analytic curve, where* I *is an interval. For* $k = 0, 1, 2, \ldots$ *or* $k = \infty$ *the class of functions* $f \in C^k(\gamma)$ *which are nowhere real analytic is a dense and* G_δ *subset of* $C^k(\gamma)$.

Remark 4.12. The fact that the class of functions $f \in C^\infty([0, 1])$ which are nowhere real analytic is itself a dense and G_δ subset of $C^\infty([0, 1])$ strengthens the result [2], where it is only proven that this class contains a dense and G_δ subset of $C^\infty([0, 1])$.

Proposition 4.13. *Let* $\gamma : I \to \mathbb{C}$ *be an analytic curve defined on an interval* $I \subset \mathbb{R}$. *Let* $\gamma^* = \gamma(I)$ *and* $\Phi : \gamma^* \to \mathbb{C}$ *be a homeomorphism of* γ^* *on*

$\Phi(\gamma^*) \subset \mathbb{C}$. *Let* $\delta = \Phi \circ \gamma : I \to \mathbb{C}$. *Then the set of functions* $f \in C^k(\delta), k \in \{0, 1, 2, ...\} \cup \{\infty\}$ *which are nowhere analytic is a* G_δ *and dense subset of* $C^k(\delta)$.

Proof. The map $S : C^k(\gamma) \to C^k(\delta)$ defined by $S(g) = g \circ \Phi^{-1}$, $g \in C^k(\gamma)$ is an isometry onto. Also a function $g \in C^k(\gamma)$ is nowhere analytic if and only if $S(g)$ is nowhere analytic. Theorem 4.11 combined with the above facts yields the result. □

Corollary 4.14. *Let* $J \subset \mathbb{R}$ *be an interval and* $\gamma(t) = t$, $t \in J$ *or* $J = \mathbb{R}$ *and* $\gamma(t) = e^{it}$, $t \in \mathbb{R}$. *Let* X *denote the image of* γ, *that is* $X = J$ *or* X *is the unit circle, respectively. Let* $\Phi : X \to \mathbb{C}$ *be a homeomorphism of* X *on* $\Phi(X) \subset \mathbb{C}$ *and* $\delta = \Phi \circ \gamma$. *Then the set of functions* $f \in C^k(\delta), k \in \{0, 1, 2, ...\} \cup \{\infty\}$ *which are nowhere analytic is a* G_δ *and dense subset of* $C^k(\delta)$.

Proof. The curve γ is an analytic curve defined on an interval. The result follows from Proposition 4.13. □

Remark 4.15. According to Corollary 4.14 for any Jordan curve δ or Jordan arc δ with a suitable parametrization generically on $C^k(\delta)$, $k \in \{0, 1, 2, ...\} \cup \{\infty\}$ every function is nowhere analytic. In fact this holds for all parametrizations of δ^* and the spaces $C^k(\delta)$ are the same for all parametrizations so that δ is a homeomorphism between the unit circle T or $[0, 1]$ and δ^* (see preliminaries).

5 Extendability of functions on domains of finite connectivity

We start this section with the following general fact.

Proposition 5.1. *Let* $n \in \{1, 2, ...\}$. *Let also* $X_1, ..., X_n$ *be complete metric spaces and* $A_1, ...A_n$ *dense and* G_δ *subsets of* $X_1, ..., X_n$, *respectively. Then the space* $X_1 \times ... \times X_n$, *endowed with the product topology, is a complete metric space and* $A_1 \times ... \times A_n$ *is a dense and* G_δ *subset of* $X_1 \times ... \times X_n$.

Proof. Obviously $X_1 \times ... \times X_n$ is a complete metric space. If $A_i = \bigcap_{k=1}^{\infty} A_{i,k}$, where $A_{i,k}$ are dense and open subsets of X_i for $i = 1, ..., n$, $k = 1, 2, ...$, then $A_1 \times ... \times A_n = \bigcap_{k=1}^{\infty} (A_{1,k} \times ... \times A_{n,k})$, where $A_{1,k} \times ... \times A_{n,k}$ are open and dense subsets of $X_1, ..., X_n$. Baire's theorem completes the proof. □

Remark 5.2. The result of Proposition 5.1 can easily be extended to infinite denumerable products, but we will not use it in the paper.

Definition 5.3. *Let $n \in \{1, 2, ...\}$. Let also $\gamma_i : I_i \to \mathbb{C}$, $i = 1, ..., n$, be continuous and locally injective curves, where I_i are intervals. We define the space $C^{p_1,...,p_n}(\gamma_1, ..., \gamma_n) = C^{p_1}(\gamma_1) \times ... \times C^{p_n}(\gamma_n)$, where $p_i \in \{0, 1, 2, ...\} \cup \{\infty\}$, for $i = 1, ..., n$. The space $C^{p_1,...,p_n}(\gamma_1, ..., \gamma_n)$ is endowed with the product topology and becomes a complete metric space.*

We can consider the above space as the class of functions f, which are defined on the disjoint union $\gamma_1^* \cup ... \cup \gamma_n^*$ of the locally injective curves $\gamma_1, ..., \gamma_n$, where $f|_{\gamma_i}$ belongs to $C^{p_i}(\gamma_i)$.

Definition 5.4. *Let $n \in \{1, 2, ...\}$. Let also $\gamma_i : I_i \to \mathbb{C}$, $i = 1, ..., n$, be locally injective curves, where I_i are intervals. A function on the disjoint union $\gamma_1^* \cup ... \cup \gamma_n^*$, $f : \gamma_1^* \cup ... \cup \gamma_n^* \to \mathbb{C}$, $f(z) = f_i(z)$ for $z \in \gamma_i^*$, belongs to the class of nowhere holomorphically extendable functions if every function f_i belongs to the class of nowhere holomorphically extendable functions defined on γ_i^*, respectively, for every $i = 1, ..., n$.*

Theorem 5.5. *Let $n \in \{1, 2, ...\}$, $p_i, q_i \in \{0, 1, ...\} \cup \{\infty\}$ such that $p_i \leq q_i$ for $i = 1, ..., n$. Let also $\gamma_i : I_i \to \mathbb{C}$, $i = 1, ..., n$, be locally injective functions belonging to $C^{q_i}(I_i)$, where I_i are intervals. The class of nowhere homolomorphically extendable functions belonging to $C^{p_1,...,p_n}(\gamma_1, ..., \gamma_n)$ is a dense and G_δ subset of $C^{p_1,...,p_n}(\gamma_1, ..., \gamma_n)$.*

Proof. Let A_i be the class of nowhere holomorphically extendable functions belonging to $C^{p_i}(\gamma_i)$ for $i = 1, ..., n$. Then the set $A_1 \times ... \times A_n$ coincides with the class of nowhere holomorphically extendable functions belonging to $C^{p_1,...,p_n}(\gamma_1, ..., \gamma_n)$. It follows from Theorem 4.9 that the sets $A_1, ..., A_n$ are dense and G_δ subsets of $C^{p_1}(\gamma_1), ..., C^{p_n}(\gamma_n)$, respectively, which combined with Proposition 5.1 implies that the class $A_1 \times ... \times A_n$ is a dense and G_δ subset of $C^{p_1,...,p_n}(\gamma_1, ..., \gamma_n)$. □

Definition 5.6. *Let $n \in \{1, 2, ...\}$. Let also $\gamma_i : I_i \to \mathbb{C}$, $i = 1, ..., n$, be locally injective curves, where I_i are intervals. A function f on the disjoint union $\gamma_1^* \cup ... \cup \gamma_n^*$, $f(z) = f_i(z)$ for $z \in \gamma_i^*$, is nowhere real analytic if the functions f_i are nowhere real analytic for $i = 1, ..., n$.*

The proof of the following theorem is similar to the proof of Theorem 5.5.

Theorem 5.7. *Let $n \in \{1, 2, ...\}$ and $p_i \in \{0, 1, ...\} \cup \{\infty\}$, $i = 1, ..., n$. Let also $\gamma_i : I_i \to \mathbb{C}$ be analytic curves, where I_i are intervals, $\Phi_i : \gamma_i^* \to \mathbb{C}$ be homeomorphisms of γ_i^* on $\Phi_i(\gamma_i) \subset \mathbb{C}$ and let $\delta_i = \Phi_i \circ \gamma_i$, $i = 1, ..., n$. The class of nowhere analytic functions $f \in C^{p_1,...,p_n}(\delta_1, ..., \delta_n)$ is a dense and G_δ subset of $C^{p_1,...,p_n}(\delta_1, ..., \delta_n)$.*

From now on, we will consider that $p_1 = p_2 = \cdots = p_n$. As we did for the spaces $C^{p_1,...,p_n}$, we will prove analogue generic results in the space $A^p(\Omega)$, where Ω is a planar domain bounded by the disjoint Jordan curves $\gamma_1, ..., \gamma_n$, $n \in \{1, 2, ...\}$. More specifically, we will define the following spaces:

Definition 5.8. *Let $p \in \{0, 1, ...\} \cup \{\infty\}$ and let Ω be a bounded domain in $\mathbb{C}$. A function f belongs to the class $A^p(\Omega)$ if it is holomorphic on Ω and every derivative f^j can be continuously extended on $\overline{\Omega}$, for all $j \in \{0, 1, ...\}$, $j \leq p$. The space $A^p(\Omega)$ is endowed with the topology of uniform convergence on $\overline{\Omega}$ of every derivative $f^{(j)}$ for all $j \in \{0, 1, ...\}$, $j \leq p$ and becomes a complete metric space.*

Remark 5.9. In particular cases it is true that $A^p(\Omega)$ is included in $C^p(\partial\Omega)$ as a closed subset. We will not examine now under which more general sufficient conditions this remains true.

Remark 5.10. If Ω is arbitrary open set in $\mathbb{C}$ (probably unbounded), then a holomorphic function $f : \Omega \to \mathbb{C}$ belongs to the class $A^p(\Omega)$ if for every $j \in \{0, 1, 2, \cdots\}, j \leq p$ the derivative $f^{(j)}$ has a continuous extension from Ω to its closure $\overline{\Omega}$ in $\mathbb{C}$. The topology of $A^p(\Omega)$ is defined by the seminorms $\sup_{z \in \overline{\Omega}, |z| \leq n} |f^{(l)}(z)|, l \in \{0, 1, 2, \cdots\}, l \leq p, n \in \mathbb{N}$

Definition 5.11. *Let Ω be a bounded domain in $\mathbb{C}$ defined by a finite number of disjoint Jordan curves. Let also $z_0 \in \partial\Omega$. A continuous function $f : \overline{\Omega} \to \mathbb{C}$ belongs to the class of non-holomorphically extendable at z_0 functions in the sense of Riemann surfaces if there do not exist open disks $D(z_0, r)$, $r > 0$ and $D(z_1, d)$, $z_1 \in D(z_0, r) \cap \Omega$, $d > 0$ such that $D(z_1, d) \subseteq D(z_0, r) \cap \Omega$, and a holomorphic function $F : D(z_0, r) \to \mathbb{C}$ such that $F|_{D(z_1,d)} = f|_{D(z_1,d)}$.*

Theorem 5.12. *Let $p \in \{0, 1, ...\} \cup \{\infty\}$ and let Ω be a bounded domain in $\mathbb{C}$ defined by a finite number of disjoint Jordan curves. Let also $z_0 \in \partial\Omega$. The class of non-holomorphically extendable at z_0 functions of $A^p(\Omega)$ in the sense of Riemann surfaces is a dense and G_δ subset of $A^p(\Omega)$.*

Proof. Let $M > 0$, $r > 0$, $z_1 \in D(z_0, r) \cap \Omega$ and $d > 0$ such that $D(z_1, d) \subset D(z_0, r) \cap \Omega$. Let also $A(p, \Omega, z_0, r, z_1, d, M)$ be the set of $A^p(\Omega)$ functions f for which there exists a holomorphic function F on $D(z_0, r)$, such that $|F(z)| \leq M$ for every $z \in D(z_0, r)$ and $F|_{D(z_1,d)} = f|_{D(z_1,d)}$. We will first show that the class $A(p, \Omega, z_0, r, z_1, d, M)$ is a closed subset of $A^p(\Omega)$ with empty interior.

Let $(f_n)_{n\geq 1}$ be a sequence in $A(p, \Omega, z_0, r, z_1, d, M)$ converging in the topology of $A^p(\Omega)$ to a function f of $A^p(\Omega)$. This implies that f_n converges uniformly on $\overline{\Omega}$ to f. Then, for $n = 1, 2, \ldots$ there are holomorphic functions $F_n : D(z_0, r) \to \mathbb{C}$ bounded by M such that $F_n|_{D(z_1,d)} = f_n|_{D(z_1,d)}$. By Montel's theorem, there exists a subsequence (F_{k_n}) of (F_n) which converges uniformly on the compact subsets of $D(z_0, r)$ to a function F which is holomorphic and bounded by M on $D(z_0, r)$. Because $F_{k_n} \to f$ at $D(z_1, d)$ we have that $F|_{D(z_1,d)} = f|_{D(z_1,d)}$ and so $f \in A(p, \Omega, z_0, r, z_1, d, M)$. Therefore, $A(p, \Omega, z_0, r, z_1, d, M)$ is a closed subset of $A^p(\Omega)$.

In addition, if $A(p, \Omega, z_0, r, z_1, d, M)$ has not empty interior, then there exist $f \in A(p, \Omega, z_0, r, z_1, d, M)$, $l \in \{0, 1, 2, ...\}$ and $\epsilon > 0$ such that

$$\begin{aligned} \{g \in A^p(\Omega) : \sup_{z\in\overline{\Omega}} \left|f^{(j)}(z) - g^{(j)}(z)\right| < \epsilon, \\ 0 \leq j \leq l\} \subset A(p, \Omega, z_0, r, z_1, d, M). \end{aligned}$$

We choose $w \in D(z_0, r)\setminus\overline{\Omega}$ and $0 < \delta < \epsilon \min\{\inf_{z\in\overline{\Omega}} |z-w|, \inf_{z\in\overline{\Omega}} |z-w|^2, ..., \frac{1}{l!} \inf_{z\in\overline{\Omega}} |z-w|^{l+1}\}$. We notice that this is possible, since $w \notin \overline{\Omega}$. The function $h(z) = f(z) + \dfrac{\delta}{2(z-w)}$ belongs to $A(p, \Omega, z_0, r, z_1, d, M)$ and therefore it has a holomorphic and bounded extension H on $D(z_0, r)$, such that $H|_{D(z_1,d)} = h|_{D(z_1,d)}$. However, there exists a holomorphic function $F : D(z_0, r) \to \mathbb{C}$ which coincides with f on $D(z_1, d)$. By analytic continuation $H(z) = F(z) + \dfrac{\delta}{2(z-w)}$ for $z \in D(z_0, r) \setminus \{z_0\}$, since they are equal on $D(z_1, d)$. As a result H is not bounded at $D(z_0, r)$ which is a contradiction. Thus, $A(p, \Omega, z_0, r, z_1, d, M)$ has empty interior.

Next, let us consider B be the set of (r, z, d, M), where $r = \dfrac{1}{n}$, $d = \dfrac{1}{m}$ for some $n, m \in \{1, 2, ...\}$ for which there exists $z \in \mathbb{Q} + i\mathbb{Q}$ such that $D(z, d) \subset D(z_0, r) \cap \Omega$, and $M \in \{1, 2, ...\}$. Then B is countable and thus there exist a sequence (b_n) such that $B = \{b_n : n \in \{1, 2, ...\}\}$. Then the class of nowhere holomorphically extendable functions of $A^p(\Omega)$ coincides with the set

$$\bigcap_{n=1}^{\infty} \left(A^p(\Omega) \setminus A\left(p, \Omega, z_0, b_n\right)\right),$$

because every holomorphic function on $D(z_0, r)$ becomes bounded if we restrict it on $D(z_0, r')$ for $r' < r$. Thus, according to Baire's theorem the class of non-holomorphically extendable at z_0 functions of $A^p(\Omega)$ is a dense and G_δ subset of $A^p(\Omega)$. □

Definition 5.13. *Let Ω be a bounded domain in $\mathbb{C}$ defined by a finite number of disjoint Jordan curves. A continuous function $f : \overline{\Omega} \to \mathbb{C}$ belongs to the class of nowhere holomorphically extendable functions in the sense of Riemann surfaces if for every $z_0 \in \partial\Omega$, f belongs to the class of non-holomorphically extendable at z_0 functions in the sense of Riemann surfaces.*

Theorem 5.14. *Let $p \in \{0, 1, ...\} \cup \{\infty\}$ and let Ω be a bounded domain in $\mathbb{C}$ defined by a finite number of disjoint Jordan curves. The class of nowhere holomorphically extendable functions of $A^p(\Omega)$ in the sense of Riemann surfaces is a dense and G_δ subset of $A^p(\Omega)$.*

Proof. Let z_l, $l = 1, ...$ be a dense sequence of $\partial\Omega$. If $A(z_l)$ is the class of non-holomorphically extendable at z_l functions of $A^p(\Omega)$ in the sense of Riemann surfaces, then, from Theorem 5.12, the set $A(z_l)$ is a dense and G_δ subset of $A^p(\Omega)$. The set $\bigcap_{l=1}^{\infty} A(z_l)$ coincides with the class of nowhere holomorphically extendable functions of $A^p(\Omega)$ in the sense of Riemann surfaces and from Baire's theorem is a dense and G_δ subset of $A^p(\Omega)$. □

Remark 5.15. In [8] it has been also proved that the class of nowhere holomorphically extendable functions of $A^\infty(\Omega)$ in the sense of Riemann surfaces is a dense and G_δ subset of $A^\infty(\Omega)$. The method in [8] comes from the theory of Universal Taylor Series and is different from the method in the present paper.

Now we will examine a different kind of extendability.

Definition 5.16. *Let Ω be a bounded domain in $\mathbb{C}$ defined by a finite number of disjoint Jordan curves $\gamma_1, \cdots, \gamma_n$. Let also $z_0 \in \partial\Omega$. A continuous function $f : \overline{\Omega} \to \mathbb{C}$ belongs to the class of non-holomorphically extendable* *at z_0 functions if there exist no pair of an open disk $D(z_0, r)$, $r > 0$ and a holomorphic function $F : D(z_0, r) \to \mathbb{C}$ such that $F(z) = f(z)$ for every $z \in D(z_0, r) \cap \partial\Omega$. Otherwise we will say that f is holomorphically extendable at z_0.*

Remark 5.17. If $\gamma_i : I_i \to \mathbb{C}$, $I_i = [a_i, b_i]$, $a_i < b_i$, is continuous and $\gamma_i(x) = \gamma_i(y)$ for $x, y \in [a_i, b_i]$ if and only if $x = y$ or $x, y\{a_i, b_i\}$, then we observe that a function f belongs to the class of non-holomorphically extendable at $z_0 = \gamma_i(t_0), t_0 \in I_i$ of Definition 5.16 if and only if there are no open disk $D(z_0, r), r > 0$ and $\eta > 0$ and a holomorphic function $F : D(z_0, r) \to \mathbb{C}$, such that $\gamma_i((t_0 - \eta, t_0 + \eta) \cap I_i) \subset D(z_0, r)$ and $F(\gamma_i(t)) = f(\gamma_i(t))$ for all $t \in (t_0 - \eta, t_0 + \eta) \cap I_i$. This holds true because of the following observations:
1)For a constant $\eta > 0$ we can find $r > 0$ such that $D(z_0, r) \cap \gamma_i^* \subseteq \gamma_i((t_0 - \eta, t_0 + \eta) \cap I_i)$. This follows from the fact that the disjoint compact sets $\gamma_i[I_i \setminus (t_0 - \eta, t_0 + \eta)]$ and $\{z_0\}$ have a strictly positive distance.
2)For a constant $r > 0$ we can find $\eta > 0$ such that $\gamma_i((t_0 - \eta, t_0 + \eta) \cap I_i) \subset D(z_0, r)$, because of the continuity of the map γ_i.

Theorem 5.18. *Let $p \in \{0, 1, ...\} \cup \{\infty\}$ and let Ω be a bounded domain in $\mathbb{C}$ defined by a finite number of disjoint Jordan curves. Let also $z_0 \in \partial\Omega$. The class of non-holomorphically extendable at z_0 functions of $A^p(\Omega)$ is a dense and G_δ subset of $A^p(\Omega)$.*

Proof. Let $M > 0$, $r > 0$ and $A(p, \Omega, z_0, r, M)$ be the class of functions $f \in A^p(\Omega)$ for which there exist a holomorphic function $F : D(z_0, r) \to \mathbb{C}$ such that $F|_{D(z_0,r)\cap\partial\Omega} = f|_{D(z_0,r)\cap\partial\Omega}$ and $|F(z)| \leq M$ for $z \in D(z_0, r)$. We will show that this class is a closed subset of $A^p(\Omega)$ with empty interior.

Similarly to the proof of Lemma 4.1 the class $A(p, \Omega, z_0, r, M)$ is a closed subset of $A^p(\Omega)$.

If $A(p, \Omega, z_0, r, M)$ has not empty interior, then there exist a function f in the interior of $A(p, \Omega, z_0, r, M)$, a number $b \in \{0, 1, 2, ...\}$ and $\delta > 0$ such that

$$\{g \in A^p(\Omega) : \sup_{z\in\overline{\Omega}} \left|f^{(j)}(z) - g^{(j)}(z)\right| < \delta,$$
$$0 \leq j \leq b\} \subset A(p, \Omega, z_0, r, M).$$

We choose $w \in D(z_0, r) \setminus \overline{\Omega}$ and $0 < a < \delta \min\{\inf_{z\in\overline{\Omega}} |z-w|, \inf_{z\in\overline{\Omega}} |z-w|^2, ..., \frac{1}{b!} \inf_{z\in\overline{\Omega}} |z-w|^{b+1}\}$. This is possible because $w \notin \overline{\Omega}$. Then, similarly to the proof of Theorem 4.7 we are led to a contradiction. Thus $A(p,\Omega,z_0,r,M)$ has empty interior.

The set

$$\bigcap_{M=1}^{\infty}\bigcap_{n=1}^{\infty}\left(A^p(\Omega)\setminus A\left(p,\Omega,z_0,\frac{1}{n},M\right)\right)$$

coincides with the class of non-holomorphically extendable at z_0 functions of $A^p(\Omega)$ and Baire's theorem implies that this set is a dense and G_δ subset of $A^p(\Omega)$. □

Definition 5.19. *Let Ω be a bounded domain in $\mathbb{C}$ defined by a finite number of disjoint Jordan curves. A continuous function $f:\overline{\Omega}\to\mathbb{C}$ belongs to the class of nowhere holomorphically extendable functions if for every $z_0\in\partial\Omega$, f belongs to the class of non-holomorphically extendable at z_0 functions.*

Theorem 5.20. *Let $p\in\{0,1,...\}\cup\{\infty\}$ and let Ω be a bounded domain in $\mathbb{C}$ defined by a finite number of disjoint Jordan curves. The class of nowhere holomorphically extendable functions of $A^p(\Omega)$ is a dense and G_δ subset of $A^p(\Omega)$.*

Proof. The proof is similar to the proof of Theorem 4.9, taking into account the statement of Theorem 5.18. □

Remark 5.21. If the continuous analytic capacity of the boundary of Ω is zero, then Definition 5.11 implies Definition 5.16.

Indeed, let Ω be a bounded domain in $\mathbb{C}$ defined by a finite number of disjoint Jordan curves, such that the continuous analytic capacity of $\partial\Omega$ is zero, and let $z_0\in\partial\Omega$. Let f be a continuous function in $\overline{\Omega}$ which does not belong to the class of Definition 5.16; That is there exist a pair of an open disk $D(z_0,r)$, $r>0$ and a holomorphic function $F:D(z_0,r)\to\mathbb{C}$ such that $F(z)=f(z)$ for every $z\in D(z_0,r)\cap\partial\Omega$. We consider the function $G:D(z_0,r)\to\mathbb{C}$

$$G(z)=\begin{cases} F(z), & z\in D(z_0,r)\setminus\Omega \\ f(z), & z\in D(z_0,r)\cap\Omega\end{cases}$$

Then G is continuous on $D(z_0,r)$ and holomorphic on $D(z_0,r)\setminus\partial\Omega$. But the continuous analytic capacity of $\partial\Omega$ is zero. From Theorem 3.6, G is holomorphic on $D(z_0,r)$ and, since the set $D(z_0,r)\cap\Omega$ is an open subset of $\mathbb{C}$, there exist $z_1\in D(z_0,r)\cap\Omega$ and $d>0$ such that $D(z_1,d)\subset D(z_0,r)\cap\Omega$. Obviously G coincides with f on $D(z_1,d)$ and thus f does not belong to the class of Definition 5.11.

Now, as in section 4, we will associate the phenomenon of non-extendability with that of real analyticity on the spaces $A^p(\Omega)$.

Definition 5.22. *Let* $n \in \{1, 2, ...\}$ *and let* Ω *be a bounded domain in* $\mathbb{C}$ *defined by disjoint Jordan curves* $\gamma_1, ..., \gamma_n$*. A function* $f : \overline{\Omega} \to \mathbb{C}$ *is real analytic at* $(t_0, \gamma_i(t_0))$*,* $\gamma_i(t_0) \in \gamma_i^*$*,* $i \in \{1, 2, ..., n\}$ *if* $f|_{\gamma_i}$ *is real analytic at* $(t_0, \gamma_i(t_0))$*.*

At this point we can observe that if Ω is a bounded domain in $\mathbb{C}$ defined by disjoint Jordan curves $\gamma_1, ..., \gamma_n$ and $f \in A^p(\Omega)$, then the analogous of Proposition 2.9 under the above assumptions holds true, since nothing essential changes in its proof. So, we have the following proposition:

Proposition 5.23. *Let* $p \in \{0, 1, ...\} \cup \{\infty\}$*,* $n \in \{0, 1, ...\}$ *and let* Ω *be a bounded domain in* $\mathbb{C}$ *defined by disjoint analytic Jordan curves* $\gamma_1, ..., \gamma_n$*. A continuous function* $f : \overline{\Omega} \to \mathbb{C}$ *is real analytic at* $(t_0, \gamma_i(t_0))$*,* $\gamma_i(t_0) \in \gamma_i^*$*,* $i \in \{1, 2, ..., n\}$ *if and only if is holomorphically extendable at* $\gamma_i(t_0)$*.*

Definition 5.24. *Let* $p \in \{0, 1, ...\} \cup \{\infty\}$*,* $n \in \{0, 1, ...\}$ *and let* Ω *be a bounded domain in* $\mathbb{C}$ *defined by disjoint Jordan curves* $\gamma_1, ..., \gamma_n$*. A function* $f \in A^p(\Omega)$ *is nowhere real analytic if there exist no* $i \in \{1, 2, ..., n\}$ *and* $\gamma_i(t_0) \in \partial\Omega$ *such that* f *is real analytic at* $(t_0, \gamma_i(t_0))$*.*

Now, combining Proposition 5.23 with Theorem 5.20, we have the following theorem:

Theorem 5.25. *Let* $p \in \{0, 1, ...\} \cup \{\infty\}$ *and let* Ω *be a bounded domain in* $\mathbb{C}$ *defined by a finite number of disjoint analytic Jordan curves. The class of nowhere real analytic functions of* $A^p(\Omega)$ *is a dense and* G_δ *subset of* $A^p(\Omega)$*.*

Remark 5.26. We recall that for an analytic Jordan curve γ defined on $[0, 1]$ there exist $0 < r < 1 < R$ and a holomorphic injective function $\Phi : D(0, r, R) \to \mathbb{C}$, such that $\gamma(t) = \Phi(e^{it})$, where $D(0, r, R) = \{z \in \mathbb{C} : r < |z| < R\}$. This yields a natural parametrization of the curve γ^*; the parameter t is called a conformal parameter for the curve γ^*. Theorem 5.25 holds if each of the Jordan curves $\gamma_1, ..., \gamma_n$ is parametrized by such a conformal parameter t. Naturally one asks if the same result holds for other parametrizations; for instance does Theorem 5.25 remains true if each $\gamma_1, ..., \gamma_n$ is parametrized by arc length? This was the motivation of [9] and [10], where it is proved that arc length is a global conformal parameter for any analytic curve. Thus, Theorem 5.25 remains also true if arc length is used as a parametrization for each analytic curve γ_i.

6 One sided extendability

In this section, we consider one sided extensions from a locally injective curve γ. For instance, if γ^* is homeomorphic to $[0,1]$, one can find an open disc D and an open arc J of γ^* separating D to two components D^+ and D^-. Those are Jordan domains containing in their boundaries a subarc J of γ. We will show that generically in $C^k(\gamma)$ every function h cannot be extended to a function $F : \Omega \cup J \to \mathbb{C}$ continuous on $\Omega \cup J$ and holomorphic in Ω; where $\Omega = D^+$ or $\Omega = D^-$. That is the one sided extendability is a rare phenomenon in $C^k(\gamma)$, provided that γ is of class at least C^k. In order to prove this fact we need the following lemmas, which are well known in algebraic topology. We include their elementary proofs for the purpose of completeness.

Lemma 6.1. *Let $\delta : [0,1] \to \mathbb{C}$ be a continuous and injective curve. Then the interior of δ^* in $\mathbb{C}$ is void.*

Proof. We will prove the lemma by contradiction. Suppose that there exists an open disk $W \subset \delta^*$. Let t_s be a point in the interior of $[0,1]$, such that $\delta(t_s) \in W$. Since the function δ is a homeomorphism of $[0,1]$ on $\delta^* \subset \mathbb{C}$, then $\delta|_{[0,1]\setminus\{t_s\}}$ is a homemorphism of $[0,1] \setminus \{t_s\}$ on $\delta^* \setminus \{\delta(t_s)\} \subset \mathbb{C}$. But the set $[0,1] \setminus \{t_s\}$ has two connected components and thus the set $\delta^* \setminus \{\delta(t_s)\}$ has two connected components V_1, V_2. Both V_1, V_2 intersect W, because we can find points of both V_1, V_2 arbitrary close to the point $\delta(t_s)$ of the open set W. Therefore, $W \cap V_1 \neq \emptyset$, $W \cap V_2 \neq \emptyset$ and $W \setminus \{\delta(t_s)\} = (W \cap V_1) \cup (W \cap V_2)$. Also, the sets V_1, V_2 are closed in the relative topology of $\delta^* \setminus \{\delta(t_s)\} \supset W \setminus \{\delta(t_s)\}$. It follows that the sets $W \cap V_1, W \cap V_2$ are closed in the relative topology of $W \setminus \{\delta(t_s)\}$. Consequently, the set $W \setminus \{\delta(t_s)\}$ is not connected, which is absurd, since it is an open disk without one of its interior points. Thus, the interior of δ^* in $\mathbb{C}$ is void. $\square$

Proposition 6.2. *Let $I \subset \mathbb{R}$ be an interval. Let also $\gamma : I \to \mathbb{C}$ be a continuous and locally injective curve. Then the interior of γ^* in $\mathbb{C}$ is void .*

Proof. Let $t_i \in I$ be such that $\gamma|_{[t_i,t_{i+1}]}$ is injective, where i varies on a finite or infinite denumerable set $S \subset \mathbb{Z}$ and

$$\bigcup_{n \in S} [t_n, t_{n+1}] = I.$$

From Lemma 6.1, the interior of every $\gamma^*_{[t_i,t_{i+1}]}$ is void in $\mathbb{C}$. Also, the set

$$\bigcup_{n\in S}\gamma^*_{[t_n,t_{n+1}]}$$

coincides with γ^* and Baire's theorem implies that the interior of γ^* is void in $\mathbb{C}$. $\square$

Proposition 6.2 implies the following:

Corollary 6.3. *Let $\gamma : I \to \mathbb{C}$ be a continuous and locally injective curve on the interval $I \subset \mathbb{R}$. Let also Ω be a Jordan domain, such that $\partial\Omega$ contains an arc of γ^*, $\gamma([t_1,t_2])$, $t_1 < t_2$, $t_1,t_2 \in I$. Then the set $\Omega \setminus \gamma^*$ is not empty.*

Let $\gamma : I \to \mathbb{C}$ be a continuous and locally injective curve defined on the interval $I \subset \mathbb{R}$. Naturally one asks if a Jordan domain as in Corollary 6.3 exists. Now, we will construct denumerably many such Jordan domains, such that every Ω, as in Corollary 6.3, contains one of these domains.

Let $t_0 \in I^\circ$, where I° is the interior of I in $\mathbb{R}$, and $s_1, s_2 \in I \cap \mathbb{Q}$, $s_1 < t_0 < s_2$, such that the map $\gamma|_{[s_1,s_2]}$ is injective. Then, the set $\gamma([s_1,t_0])$ is compact and the point $\gamma(s_2)$ does not belong to $\gamma([s_1,t_0])$. Thus, there exists $0 < \delta = dist(\gamma(s_2), \gamma([s_1,t_0]))$. From Proposition 6.2, there exists $P \in (\mathbb{Q}+i\mathbb{Q}) \cap D(\gamma(s_2), \delta/2)$, $P \notin \gamma([s_1,s_2])$. Let $0 < a = dist(P, \gamma([s_1,s_2])) \leq |P-\gamma(s_2)| < \delta/2$. Then there exists $t_0 < r \leq s_2$, such that $|\gamma(r) - P| = a$, because, for every $t \in [s_1,t_0]$,

$$|\gamma(t) - P| \geq |\gamma(t) - \gamma(s_2)| - |\gamma(s_2) - P| > \delta - \delta/2 = \delta/2$$

Then, the segment $[P, \gamma(r)]$ intersects $\gamma([s_1,s_2])$ only at $\gamma(r)$. Similarly, if $0 < \varepsilon = dist(\gamma(s_1), \gamma([t_0,s_2]))$, there exist $Q \in (\mathbb{Q}+i\mathbb{Q}) \cap D(\gamma(s_1), \varepsilon/2)$, $Q \notin \gamma([s_1,s_2])$ and $s_1 \leq \widetilde{r} < t_0$, such that the segment $[Q, \gamma(\widetilde{r})]$ intersects $\gamma([s_1,s_2])$ only at $\gamma(\widetilde{r})$.

We distinguish two cases according to whether the segments $[P, \gamma(r)]$, $[Q, \gamma(\widetilde{r})]$ intersect or not. If the segments $[P, \gamma(r)]$, $[Q, \gamma(\widetilde{r})]$ intersect at a point w, then the union of the segments $[w, \gamma(r)]$, $[w, \gamma(\widetilde{r})]$ and $\gamma[\widetilde{r}, r]$ is the image of a Jordan curve, the interior of which is one of the desired Jordan domains. If the segments $[P, \gamma(r)]$, $[Q, \gamma(\widetilde{r})]$ do not intersect, then we consider a simple polygonal line (that is without self intersections) in $\mathbb{C} \setminus \gamma([s_1,s_2])$, which connects P, Q, the vertices of which belong to $\mathbb{Q}+i\mathbb{Q}$. This is possible, since the set $\mathbb{C}\setminus\gamma([s_1,s_2])$ is a domain ([11]). We consider one of the connected components of this simple polygonal line minus the segments $[P, \gamma(r)]$, $[Q, \gamma(\widetilde{r})]$, such that the closure of this connected component is a simple polygonal line connecting two points $z_1 \in [P, \gamma(r)]$, $z_2 \in [Q, \gamma(\widetilde{r})]$. Then, the union of the last simple polygonal line with the segments $[z_1, \gamma(r)]$, $[z_2, \gamma(\widetilde{r})]$ and the arc of γ^*, $\gamma[\widetilde{r}, r]$ is the image of a Jordan curve, the interior of which is one of the desired Jordan domains. We notice that $t_0 \in [\widetilde{r}, r]$ and that the constructed Jordan domains are denumerably many.

Proposition 6.4. *Let $\gamma : I \to \mathbb{C}$ be a locally injective map of class $C^l(I)$, $l \in \{0, 1, 2, ...\} \cup \{\infty\}$ on the interval $I \subset \mathbb{R}$. Let also Ω be a Jordan domain, such that $\partial\Omega$ contains an arc of γ^*, $\gamma([t_1, t_2])$, $t_1 < t_2$, $t_1, t_2 \in I$ and $k \in \{0, 1, 2, ...\} \cup \{\infty\}$, $k \leq l$. Let $0 < M < \infty$. The set of functions $f \in C^k(\gamma)$ for which there exists a continuous function $F : \Omega \cup \gamma((t_1, t_2))$, $||F||_\infty \leq M$, such that F is holomorphic on Ω and $F|_{\gamma((t_1,t_2))} = f|_{\gamma((t_1,t_2))}$, is a closed subset of $C^k(\gamma)$ with empty interior.*

Proof. Let Ψ be a homeomorphism of $D \cup J \subset \mathbb{C}$ on $\Omega \cup \gamma(t_1, t_2)$, which is also holomorphic on Ω, where $J = \{e^{it} : 0 < t < 1\}$. This is possible because of the Caratheodory-Osgood theorem. Let also $A(k, \Omega, M)$ be the set of functions $f \in C^k(\gamma)$ for which there exists a continuous function $F : \Omega \cup \gamma((t_1, t_2)) \to \mathbb{C}$, $||F||_\infty \leq M$, such that F is holomorphic on Ω and $F|_{\gamma((t_1,t_2))} = f|_{\gamma((t_1,t_2))}$.

First, we will prove that $A(k, \Omega, M)$ is a closed subset of $C^k(\gamma)$. Let $(f_n)_{n\geq1}$ be a sequence in $A(k, \Omega, M)$ converging in the topology of $C^k(\gamma)$ to a function $f \in C^l(\gamma)$. This implies that f_n converges uniformly on the compact subsets of γ^* to f. Then, for $n = 1, 2, \ldots$ there exist continuous functions $F_n : \Omega \cup \gamma((t_1, t_2)) \to \mathbb{C}$, $||F_n||_\infty \leq M$, such that F_n are holomorphic on Ω and $F_n|_{\gamma((t_1,t_2))} = f_n|_{\gamma((t_1,t_2))}$. If $G_n = F_n \circ \Psi$, $g_n = f_n \circ \Psi$ and $g = f \circ \Psi$ for $n = 0, 1, 2, \ldots$, it follows that g_n converges uniformly on the compact subsets of J to g. Also, the functions G_n are holomorphic and bounded by M on D. By Montel's theorem, there exists a subsequence of (G_n), (G_{k_n}) which converges uniformly on the compact subsets of D to a function G which is holomorphic and bounded by M on D. Without loss of generality, we assume that $(G_n) = (G_{k_n})$. Now, it is sufficient to prove that for any circular sector K, which has boundary $[0, e^{ia}] \cup [0, e^{ib}] \cup \{e^{it} : a \leq t \leq b\}$, $0 < a < b < 1$, the sequence (G_n) converges uniformly on K, because then the limit of (G_n), which is equal to g at the arc J and equal to G on the remaining part of the circular sector K, will be a continuous function. In order to do so, we will prove that (G_n) is a uniformly Cauchy sequence on K. Because each

G_n is a bounded holomorphic function on D, we know that for every n, the radial limits of G_n exist almost everywhere on the unit circle and so we can consider the respective functions g_n defined almost everywhere on the unit circle which are extensions of the previous g_n. These g_n are also bounded by M.
Let $\varepsilon > 0$ be a positive number. For the Poisson kernel P_r, $0 \leq r < 1$ and for every $n = 0, 1, 2 \dots$, it holds that $G_n(re^{i\theta}) = \dfrac{1}{2\pi} \int\limits_{-\pi}^{\pi} P_r(t) g_n(\theta - t) dt$.
We choose $0 < \delta < min\{1 - b, a\}$. There exists $0 < r_0 < 1$ such that $\sup\limits_{\delta \leq |t| \leq \pi} P_r(t) < \frac{\varepsilon}{8M}$ for every $r \in [r_0, 1)$. Then, (G_n) is a uniformly Cauchy sequence on $K \cap \{z \in \mathbb{C} : |z| \leq r_0\}$, and thus, there exists n_1 such that for every $n, m \geq n_1$,

$$\sup_{z \in K \cap \{z \in \mathbb{C} : |z| \leq r_0\}} |G_n(z) - G_m(z)| < \frac{\varepsilon}{2} \qquad (1)$$

In addition, because g_n converges uniformly to g on J there exists n_2, such that for every $n, m \geq n_2$, $\sup\limits_{z \in J} |g_n(z) - g_m(z)| < \dfrac{\varepsilon}{4}$. Consequently, for $n, m \geq \max\{n_1, n_2\}$, for $\theta \in [a, b]$ and for $1 > r > r_0$

$$|G_n(re^{i\theta}) - G_m(re^{i\theta})| \leq \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} P_r(t) |g_n(\theta - t) - g_m(\theta - t)| dt =$$

$$\frac{1}{2\pi} \int\limits_{-\delta}^{\delta} P_r(t) |g_n(\theta - t) - g_m(\theta - t)| dt +$$

$$+ \frac{1}{2\pi} \int\limits_{\delta \leq |t| \leq \pi} P_r(t) |g_n(\theta - t) - g_m(\theta - t)| dt \leq$$

$$\frac{1}{2\pi} \int\limits_{-\delta}^{\delta} P_r(t) \sup_{z \in J} |g_n(z) - g_m(z)| dt + \frac{\sup\limits_{\delta \leq |t| \leq \pi} P_r(t)}{2\pi} \int\limits_{\delta \leq |t| \leq \pi} 2M dt \leq$$

$$\frac{\varepsilon}{4} \frac{1}{2\pi} \int\limits_{-\delta}^{\delta} P_r(t) dt + \frac{\varepsilon}{16M\pi} \int\limits_{\delta \leq |t| \leq \pi} 2M dt \leq$$

$$\frac{\varepsilon}{4}\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(t)dt + \frac{\varepsilon}{4} = \frac{\varepsilon}{2} \qquad (2)$$

By the continuity of the functions G_n on $D \cup J$, making $r \to 1^-$, we find that

$$|G_n(e^{i\theta}) - G_m(e^{i\theta})| \leq \frac{\varepsilon}{2}, \qquad (3)$$

for all $\theta \in [a, b]$. Therefore, from (1), (2) and (3), it follows that (G_n) is a uniformly Cauchy sequence on the circular sector K and thus the set $A(k, \Omega, M)$ is a closed subset of $C^k(\gamma)$.

If $A(k, \Omega, M)$ has not empty interior, then there exists a function f in the interior of $A(k, \Omega, M)$, a compact set $L \subset I$ and $\delta > 0$ such that

$$\{g \in C^k(\gamma) : \sup_{t \in L} \left|(f \circ \gamma)^{(j)}(t) - (g \circ \gamma)^{(j)}(t)\right| < \delta,$$
$$0 \leq j \leq b\} \subset A(k, \Omega, M).$$

From Corollary 6.3, we can find a $w \in \Omega \setminus \gamma^*$. We choose $0 < a < \delta \min\{\inf_{t \in K} |\gamma(t) - w|, \inf_{t \in K} |\gamma(t) - w|^2, ..., \frac{1}{b!} \inf_{t \in K} |\gamma(t) - w|^{b+1}\}$. This is possible because $w \notin \gamma^*$ and $\gamma(K) \subset \gamma^*$. The function $h(\gamma(t)) = f(\gamma(t)) + \frac{a}{2(\gamma(t) - w)}$, $t \in I$ belongs to $A(k, \Omega, M)$ and therefore has a continuous and bounded extension H on $\Omega \cup \gamma((t_1, t_2))$ with $H|_{\gamma((t_1,t_2))} = h|_{\gamma((t_1,t_2))}$ which is holomorphic on Ω. Then, the function $H \circ \Psi$ is continuous and bounded on $D \cup J$ and holomorhic on D. We can easily see that $H(\Psi(z)) = F(\Psi(z)) + \frac{a}{2(\Psi(z) - w)}$ for $z \in D \setminus \{\Psi^{-1}(w)\}$. Indeed, let $\Phi(z) = H(\Psi(z)) - F(\Psi(z)) - \frac{a}{2(\Psi(z) - w)}$. Then, $\Phi|_J = 0$ and by Schwarz Reflection Principle Φ is extended holomorphically on

$$(D \cup J \cup (\mathbb{C} \setminus \overline{D})) \setminus \{\Psi^{-1}(w), \frac{1}{\overline{\Psi^{-1}(w)}}\}.$$

Therefore, because $\Phi = 0$ on J, by analytic continuation, $H(\Psi(z)) - F(\Psi(z)) - \frac{a}{2(\Psi(z) - w)} = 0$ on $(D \cup J) \setminus \{\Psi^{-1}(w)\}$. As a result $H \circ \Psi$ is not bounded on D which is absurd. Thus, $A(k, \Omega, M)$ has empty interior. $\square$

Definition 6.5. *Let $\gamma : I \to \mathbb{C}$ be a locally injective map on the interval $I \subset \mathbb{R}$. Let also $t_0 \in I^\circ$, where I° is the interior of I in $\mathbb{R}$. A function*

$f : \gamma^* \to \mathbb{C}$ is non one sided holomorphically extendable at $(t_0, \gamma(t_0))$ if there exists no pair of a Jordan domain Ω, such that $\partial\Omega$ contains an arc of γ^*, $\gamma([t_1, t_2])$, $t_1 < t_0 < t_2$, $t_1, t_2 \in I$ and a continuous function $F : \Omega \cup \gamma((t_1, t_2))$, which is holomorphic on Ω and $F|_{\gamma((t_1,t_2))} = f|_{\gamma((t_1,t_2))}$.

Theorem 6.6. *Let $\gamma : I \to \mathbb{C}$ be a locally injective map of class $C^l(I)$, $l \in \{0, 1, 2, ...\} \cup \{\infty\}$ on the interval $I \subset \mathbb{R}$ and $k \in \{0, 1, 2, ...\} \cup \{\infty\}$, $k \leq l$. Let also $t_0 \in I^\circ$, where I° is the interior of I in $\mathbb{R}$. The class of non one sided holomorphically extendable at $(t_0, \gamma(t_0))$ functions of $C^k(\gamma)$ is a dense and G_δ subset of $C^k(\gamma)$.*

Proof. If Ω is a Jordan domain, such that $\partial\Omega$ contains an arc of γ^*, $\gamma([t_1, t_2])$, $t_1 < t_0 < t_2$, $t_1, t_2 \in I$, then $A(k, \Omega, M)$ denotes the set of functions $f \in C^k(\gamma)$ for which there exists a continuous and bounded by M function $F : \Omega \cup \gamma((t_1, t_2))$, which is holomorphic on Ω and $F|_{\gamma((t_1,t_2))} = f|_{\gamma((t_1,t_2))}$. Let G_n, $n = 1, 2, ...$ be the denumerably many Jordan domains constructed above, such that $s_1 < t_0 < s_2$ where $s_1, s_2 \in \mathbb{Q} \cap I$ are as in the construction of G_n and t_0 is the fixed real number in the statement of Theorem 6.6 .

From Proposition 6.4, the sets $A(k, G_n, M)$ are closed subsets of $C^k(\gamma)$ with empty interior. We will prove that the class of non one sided holomorphically extendable at $(t_0, \gamma(t_0))$ functions of $C^k(\gamma)$ coincides with the set

$$\bigcap_{n=1}^{\infty} \bigcap_{M=1}^{\infty} (C^k(\gamma) \setminus A(k, G_n, M)),$$

and thus, Baire's theorem will imply that the above set is a dense and G_δ subset of $C^k(\gamma)$.

Obviously, the set

$$\bigcap_{n=1}^{\infty} \bigcap_{M=1}^{\infty} (C^k(\gamma) \setminus A(k, G_n, M))$$

contains the class of non one sided holomorphically extendable at $(t_0, \gamma(t_0))$ functions of $C^k(\gamma)$. Reversely, let Ω be a Jordan domain whose boundary contains an arc $\gamma([t_1, t_2])$, $t_1 < t_0 < t_2$, $t_1, t_2 \in I$. Let also $f \in C^k(\gamma)$, for which there exists a continuous function $F : \Omega \cup \gamma((t_1, t_2))$, which is holomorphic on Ω and $F|_{\gamma((t_1,t_2))} = f|_{\gamma((t_1,t_2))}$. From the construction of the above Jordan domains G_n, we can find $P, Q \in \Omega \cap (\mathbb{Q} + i\mathbb{Q})$ and $t_1 < \widetilde{r} < t_0 < r < t_2$, such that the segments $[P, \gamma(r)]$, $[Q, \gamma(\widetilde{r})]$ intersect $\partial\Omega$ only at

$\gamma(r)$, $\gamma(\widetilde{r})$, respectively. The only modifications needed in order to do that are the following: we fix $t_1 < t_3 < t_0 < t_4 < t_2$, $t_3, t_4 \in \mathbb{Q}$ and consider a number smaller than the half of the distance of $\gamma(t_3)$ from the compact set $\partial\Omega \setminus \gamma(t_1, t_0)$ and a number smaller than the half of the distance of $\gamma(t_4)$ from the compact set $\partial\Omega \setminus \gamma(t_0, t_2)$. We continue as in the construction, with the only possible difference that the rational numbers t_3 and t_4 will replace s_1, s_2 in the construction of the G_n after Corollary 6.3 and, if a simple polygonal line, which connects P, Q and with vertices in $\mathbb{Q} + i\mathbb{Q}$, is needed, we consider it also in the domain Ω. This Jordan domain is one of the denumerable Jordan domains, G_{n_0}, constructed above and $\overline{G_{n_0}}$ is contained in $\Omega \cup \gamma([\widetilde{r}, r])$. It easily follows that $F|_{G_{n_0}}$ is bounded by some number $M = 1, 2, 3, ...$ and thus f belongs to $A(k, G_{n_0}, M)$. Therefore, the class of non one sided holomorphically extendable at $(t_0, \gamma(t_0))$ functions of $C^k(\gamma)$ is a subset of the set

$$\bigcap_{n=1}^{\infty} \bigcap_{M=1}^{\infty} (C^k(\gamma) \setminus A(k, G_n, M)),$$

which combined with the above completes the proof. □

Definition 6.7. *Let $\gamma : I \to \mathbb{C}$ be a locally injective map on the interval $I \subset \mathbb{R}$. A function $f : \gamma^* \to \mathbb{C}$ is nowhere one sided holomorphically extendable if there exists no pair of a Jordan domain Ω, such that $\partial\Omega$ contains an arc of γ^*, $\gamma([t_1, t_2])$, $t_1 < t_2$, $t_1, t_2 \in I$ and a continuous function $F : \Omega \cup \gamma((t_1, t_2))$, which is holomorphic on Ω and $F|_{\gamma((t_1,t_2))} = f|_{\gamma((t_1,t_2))}$.*

Theorem 6.8. *Let $\gamma : I \to \mathbb{C}$ be a locally injective map of class C^l, $l \in \{0, 1, 2, ...\} \cup \{\infty\}$ on the interval $I \subset \mathbb{R}$ and $k \in \{0, 1, 2, ...\} \cup \{\infty\}$, $k \le l$. The class of nowhere one sided holomorphically extendable functions of $C^k(\gamma)$ is a dense and G_δ subset of $C^k(\gamma)$.*

Proof. Let $t_n \in I^\circ$, $n = 1, 2, ...$, where I° is the interior of I in $\mathbb{R}$, be a dense sequence in I. Then the class of nowhere one sided holomorphically extendable functions of $C^k(\gamma)$ coincides with the intersection of the classes of non one sided holomorphically extendable at $(t_n, \gamma(t_n))$ functions of $C^k(\gamma)$, which is from Theorem 6.6 and Baire's theorem a dense and G_δ subset of $C^k(\gamma)$. □

Let $n \in \{1, 2, ...\}$ and let $\gamma_i : I_i \to \mathbb{C}$, $i = 1, ..., n$, be continuous and locally injective curves, where I_i are intervals. We recall the definition of the space $C^{p_1,...,p_n}(\gamma_1, ..., \gamma_n) = C^{p_1}(\gamma_1) \times ... \times C^{p_n}(\gamma_n)$, where $p_i \in \{0, 1, 2, ...\} \cup$

$\{\infty\}$, for $i = 1, ..., n$. Now, as we did in section 5, we will prove generic results for the spaces $C^{p_1,...,p_n}(\gamma_1, ..., \gamma_n)$.

Definition 6.9. *Let $n \in \{1, 2, ...\}$. Let also $\gamma_i : I_i \to \mathbb{C}$, $i = 1, ..., n$, be locally injective curves, where I_i are intervals and t_0 a point of some I_{i_0}. A function f on the disjoint union $\gamma_1^* \cup ... \cup \gamma_n^*$, $f(z) = f_i(z)$ for $z \in \gamma_i^*$, is non oned sided holomorphically extendable at $(t_0, \gamma_{i_0}(t_0))$ if the function f_{i_0} defined on $\gamma_{i_0}^*$ is non one sided holomorphically extendable at $(t_0, \gamma_{i_0}(t_0))$.*

Theorem 6.10. *Let $n \in \{1, 2, ...\}$, $p_i, q_i \in \{0, 1, ...\} \cup \{\infty\}$ such that $p_i \leq q_i$ for $i = 1, ..., n$. Let also $\gamma_i : I_i \to \mathbb{C}$, $i = 1, ..., n$, be locally injective functions belonging to $C^{q_i}(I_i)$, where I_i are intervals, and t_0 a point of some I_{i_0}. The class of non one sided homolomorphically extendable at $(t_0, \gamma_{i_0}(t_0))$ functions belonging to $C^{p_1,...,p_n}(\gamma_1, ..., \gamma_n)$ is a dense and G_δ subset of $C^{p_1,...,p_n}(\gamma_1, ..., \gamma_n)$.*

Proof. Similar to the proof of Theorem 5.5. □

Definition 6.11. *Let $n \in \{1, 2, ...\}$. Let also $\gamma_i : I_i \to \mathbb{C}$, $i = 1, ..., n$, be locally injective curves, where I_i are intervals. A function f on the disjoint union $\gamma_1^* \cup ... \cup \gamma_n^*$, $f(z) = f_i(z)$ for $z \in \gamma_i^*$, is nowhere one sided holomorphically extendable if the functions f_i defined on γ_i^* are nowhere one sided holomorphically extendable functions.*

The proof of the following theorem is also similar to the proof of Theorem 5.5.

Theorem 6.12. *Let $n \in \{1, 2, ...\}$, $p_i, q_i \in \{0, 1, ...\} \cup \{\infty\}$ such that $p_i \leq q_i$ for $i = 1, ..., n$. Let also $\gamma_i : I_i \to \mathbb{C}$, $i = 1, ..., n$, be locally injective functions belonging to $C^{q_i}(I_i)$, where I_i are intervals. The class of nowhere one sided homolomorphically extendable functions belonging to $C^{p_1,...,p_n}(\gamma_1, ..., \gamma_n)$ is a dense and G_δ subset of $C^{p_1,...,p_n}(\gamma_1, ..., \gamma_n)$.*

Definition 6.13. *Let $n \in \{1, 2, ...\}$ and let Ω be a bounded domain in $\mathbb{C}$ defined by disjoint Jordan curves $\gamma_i : \mathbb{R} \to \mathbb{C}$, where each γ_i is continuous and periodic, $i = 1, ..., n$. Let also $t_0 \in \mathbb{R}$. A function $f : \overline{\Omega} \to \mathbb{C}$ is non one sided holomorphically extendable at $(t_0, \gamma_{i_0}(t_0))$ outside Ω if there exists no pair of a Jordan domain $G \subset \mathbb{C} \setminus \overline{\Omega}$, such that ∂G contains a Jordan arc of $\gamma_{i_0}^*$, $\gamma_{i_0}([t_1, t_2])$, $t_1 < t_0 < t_2$, $t_1, t_2 \in \mathbb{R}$ and a continuous function $F : \Omega \cup \gamma((t_1, t_2))$, which is holomorphic on G and $F|_{\gamma_{i_0}((t_1,t_2))} = f|_{\gamma_{i_0}((t_1,t_2))}$.*

Theorem 6.14. *Let $p \in \{0, 1, ...\} \cup \{\infty\}$, $n \in \{1, 2, ...\}$ and let Ω be a bounded domain in $\mathbb{C}$ defined by disjoint Jordan curves $\gamma_i : \mathbb{R} \to \mathbb{C}$, where each γ_i is continuous and periodic, $i = 1, ..., n$. Let also $t_0 \in \mathbb{R}$. The class of non one sided holomorphically extendable at $(t_0, \gamma_{i_0}(t_0))$ outside Ω functions of $A^p(\Omega)$ is a dense and G_δ subset of $A^p(\Omega)$.*

Proof. The proof is a combination of proofs similar to those of Proposition 6.4 and Theorem 6.6. □

Definition 6.15. *Let $n \in \{1, 2, ...\}$ and let Ω be a bounded domain in $\mathbb{C}$ defined by disjoint Jordan curves $\gamma_i : \mathbb{R} \to \mathbb{C}$, , where each γ_i is continuous and periodic, $i = 1, ..., n$. A function $f : \overline{\Omega} \to \mathbb{C}$ is nowhere one sided holomorphically extendable outside Ω if f is non one sided holomorphically extendable at $(t_0, \gamma_{i_0}(t_0))$ outside Ω, for every $t_0 \in \mathbb{R}$ and $i_0 = 1, 2, ..., n$.*

Combining Theorem 6.14 for a dense in $\mathbb{R}$ sequence t_n and Baire's theorem, we obtain the following:

Theorem 6.16. *Let $p \in \{0, 1, ...\} \cup \{\infty\}$ and let Ω be a bounded domain in $\mathbb{C}$ defined by a finite number of disjoint Jordan curves. The class of nowhere one sided holomorphically extendable functions outside of Ω is a dense and G_δ subset of $A^p(\Omega)$.*

Remark 6.17. In the above statements the functions f are defined on $\overline{\Omega}$. However, for $f \in A^p(\Omega)$, by the maximal principle, there is a one to one correspondence between the function f and $f|_{\partial\Omega}$. Therefore, we could state the previous result considering $A^p(\Omega)$ restricted to $\partial\Omega$.

7 Removability of singularities in the spaces A^p and the p-continuous analytic capacities. A dichotomy result

Let Ω denote an open and bounded subset of $\mathbb{C}$ and L be a compact subset of Ω. Let also $G = \Omega \setminus L$, which is an open subset of $\mathbb{C}$, $p \in \{0, 1, ...\} \cup \{\infty\}$ and let $f_0 \in A^p(G)$. Then, there are two cases:

(i) Either there exists a function $F_0 \in A^p(\Omega)$, such that $F_0|_G = f_0$

(ii) or there exists no $F_0 \in A^p(\Omega)$, such that $F_0|_G = f_0$.

Theorem 7.1. *Let $p \in \{0, 1, ...\} \cup \{\infty\}$, let Ω denote an open and bounded subset of $\mathbb{C}$ and L be a compact subset of Ω. Let also $G = \Omega \setminus L$. Suppose that there exists a function $f_0 \in A^p(G)$ for which there exists no $F_0 \in A^p(\Omega)$, such that $F_0|_G = f_0$. Then the set of functions $f \in A^p(G)$ for which there exists no $F \in A^p(\Omega)$, such that $F|_G = f$, is an open and dense subset of $A^p(G)$.*

Proof. Let $A(p, G)$ be the set of functions $f \in A^p(G)$ for which there exists a continuous function $F \in A^p(\Omega)$, such that $F|_G = f$.

First, we will prove that the set $A(p, G)$ is a closed subset of $A^p(G)$. Let $(f_n)_{n\geq 1}$ be a sequence in $A(p, G)$ converging in the topology of $A^p(G)$ to a function $f \in A^p(G)$. This implies that f_n converges uniformly on $\overline{G}$ to f. By the maximum principle, the extensions F_n of f_n form a uniformly Cauchy sequence on $\overline{\Omega}$. Thus, the limit F of F_n on $\overline{\Omega}$ is an extension of f. Therefore, $f \in A(p, G)$ and $A(p, G)$ is a closed subset of $A^p(G)$.

Now, we will prove that the set $A(p, G)$ has empty interior in $A^p(G)$. If $A(p, G)$ has not empty in $A^p(G)$, then there exists a function f in the interior of $A(p, G)$ and there exist $l \in \{0, 1, ...\}$, $l \leq p$ and $d > 0$, such that

$$\{g \in A^p(G) : \sup_{z\in\overline{G}} |f^{(j)}(z) - g^{(j)}(z)| < d, 0 \leq j \leq l\} \subset A(p, G).$$

It is easy to see that the function f_0 is not identically equal to zero. Thus, $m = max\{\sup_{z\in\overline{G}} |{f_0}^{(j)}(z)|, j = 0, 1, ..., l\} > 0$. Then, the function $g(z) = f(z) + \dfrac{d}{2m} f_0(z)$, $z \in \overline{G}$ belongs to $A(p, G)$. Since the functions f, g belong to $A(p, G)$, there are functions $F, H \in A^p(\Omega)$, such that $F|_G = f$, $H|_G = g$. Then, the function $\dfrac{2m}{d}(H(z) - F(z))$ belongs to $A^p(\Omega)$ and is equal to f_0 in G, which contradicts our hypothesis. Thus, the set $A(p, G)$ has empty interior in $A^p(G)$.

The set of functions $f \in A^p(G)$ for which there exists no $F \in A^p(\Omega)$, such that $F|_G = f$ coincides with the set

$$A^p(G) \setminus A(p, G)$$

and the above set is an open and dense subset of $A^p(G)$. □

Remark 7.2. If the interior of L in $\mathbb{C}$ is not empty, then there always exists a function $f_0 \in A^p(G)$, for which there does not exist a function $F_0 \in A^p(\Omega)$, such that $F_0|_G = f_0|_G$. Indeed, let $w \in L^o$; then $f_0 = \dfrac{1}{z-w}$ belongs to $A^p(G)$ but it can not have an extension in $A^p(\Omega)$. Thus, the class of functions $f \in A^p(G)$ for which there exists no $F \in A^p(\Omega)$, such that $F|_G = f$ is dense and open in $A^p(G)$.

Remark 7.3. From the previous results we have a dichotomy: Either every $f \in A^p(G)$ has an extension in $A^p(\Omega)$ or generically all functions $f \in A^p(G)$ do not admit any extension in $A^p(\Omega)$. The first case holds if and only if $a_p(L) = 0$ and the second case if and only if $a_p(L) > 0$ (Theorem 3.12).

Remark 7.4. The results of this section can easily be extended to unbounded open sets $\Omega \subset \mathbb{C}$ with the only difference in the definition of the topology of $A^p(\Omega)$. The topology of $A^p(\Omega)$ is defined by the denumerable family of seminorms

$$\sup_{z\in\Omega,|z|\leq n} |f^{(l)}(z)|, l = 0, 1, 2, \cdots, n = 0, 1, 2, \cdots .$$

For $p < \infty$ and Ω unbounded $A^p(\Omega)$ is a Frechet space, while if Ω is bounded it is a Banach space.

Remark 7.5. In a similar way we can prove that if L is a compact set contained in the open set U, then either every $f \in \tilde{A}^p(U \setminus L)$ has an extension in $\tilde{A}^p(U)$ or generically every $f \in \tilde{A}^p(U \setminus L)$ does not have an extension in $\tilde{A}^p(U)$, $p \in \{0, 1, 2, \cdots\} \cup \{\infty\}$. The first horn of this dichotomy holds if and only if $\tilde{a}_p(L) = 0$ which is equivalent with the fact that the interior of L is void in $\mathbb{C}$ (Theorem 3.16).

Now, we present some local versions of the results of section 7.

Definition 7.6. *Let L be a compact subset of $\mathbb{C}$ and U be an open subset of $\mathbb{C}$, such that $L \subseteq U$. Let also $z_0 \in \partial L$. A function $f \in H(U \setminus L)$ is extendable at z_0 if there exists $r > 0$ and $F \in H(D(z_0, r))$ such that $F|_{(U\setminus L)\cap D(z_0,r)} = f|_{(U\setminus L)\cap D(z_0,r)}$. Otherwise, we say that f is not extendable at z_0.*

Below, we will use the above definition of extendability.

Proposition 7.7. *Let L be a compact subset of $\mathbb{C}$ and U be an open subset of $\mathbb{C}$, such that $L \subseteq U$. Let also M and r be positive real numbers, $p \in \{0, 1, 2, \cdots\} \cup \{\infty\}$ and $z_0 \in \partial L$. The set*

$$E_{M,p,U,L,z_0,r} = \{f \in A^p(U \setminus L) : \textit{there exists } F \in H(D(z_0, r)) \textit{ such}$$

that $F|_{(U\setminus L)\cap D(z_0,r)} = f|_{(U\setminus L)\cap D(z_0,r)}$ *and* $\| F \|_\infty = \sup_{z\in D(z_0,r)} |F(z)| \leq M\}$

is a closed subset of $A^p(U \setminus L)$*. Also, if there exists* $f_0 \in A^p(U \setminus L)$ *which is not extendable at* z_0*, then the interior of* $E_{M,p,U,L,z_0,r}$ *is void in* $A^p(U \setminus L)$*.*

Proof. We will first prove that the set $E_{M,p,U,L,z_0,r}$ is a closed subset of $A^p(U\setminus L)$. Let $(f_n)_{n\geq 1}$ be a sequence in $E_{M,p,U,L,z_0,r}$ converging in the topology of $A^p(U \setminus L)$ to a function $f \in A^p(U \setminus L)$. This implies that f_n converges uniformly on $\overline{U \setminus L}$ to f and that there exists a sequence $(F_n)_{n\geq 1}$ in $H(D(z_0, r))$ such that $F_n|_{(U\setminus L)\cap D(z_0,r)} = f_n|_{(U\setminus L)\cap D(z_0,r)}$ and $\| F \|_\infty \leq M$ for every $n \geq 1$. By Montel's theorem, there exists a subsequence of (F_n), (F_{k_n}), which converges uniformly on the compact subsets of $D(z_0, r)$ to a function F which is holomorphic and bounded by M on $D(z_0, r)$. Since F_{k_n} converges to f on $(U\setminus L)\cap D(z_0, r)$, the functions f and F are equal on $(U\setminus L)\cap D(z_0, r)$. Thus, f belongs to $E_{M,p,U,L,z_0,r}$ and $E_{M,p,U,L,z_0,r}$ is a closed subset of $A^p(U \setminus L)$.
If there exists $f_0 \in A^p(U \setminus L)$ which is not extendable at z_0, the interior of $E_{M,p,U,L,z_0,r}$ is void in $A^p(U \setminus L)$, the proof of which is similar to the proof of Theorem 7.1. □

Here, we have one more dichotomy, which is a local version of the first one.

Theorem 7.8. *Let* L *be a compact subset of* $\mathbb{C}$ *and* U *be an open subset of* $\mathbb{C}$*, such that* $L \subseteq U$ *and let* $p \in \{0, 1, 2, \cdots\} \cup \{\infty\}$*,* $z_0 \in \partial L$*. The set* $E_{p,U,L,z_0} = \bigcup_{M=1}^{\infty} \bigcup_{n=1}^{\infty} E_{M,p,U,L,z_0,\frac{1}{n}}$ *is the set of extendable functions of* $A^p(U \setminus L)$ *at* z_0*. Then*

- *(i) either every function* $f \in A^p(U \setminus L)$ *is extendable at* z_0
- *(ii) or generically all functions* $f \in A^p(U \setminus L)$ *are not extendable at* z_0*.*

Proof. If (i) is not true, then Proposition 7.7 shows that $E_{M,p,U,L,z_0,\frac{1}{n}}$ is closed with empty interior for all natural numbers $n \geq 1, M \geq 1$. Then,

$$A^p(U \setminus L) \setminus E_{p,U,L,z_0} = \bigcap_{M=1}^{\infty} \bigcap_{n=1}^{\infty} (A^p(U \setminus L) \setminus E_{M,p,U,L,z_0,\frac{1}{n}})$$

is the intersection of a countable number of open and dense subsets of $A^p(U\setminus L)$ and Baire's Theorem shows that $A^p(U\setminus L)\setminus E_{p,U,L,z_0}$, which coincides with the set of non extendable functions of $A^p(U \setminus L)$ at z_0, is a dense and G_δ subset of $A^p(U \setminus L)$. □

Now, we will compare two notions: local extendability and existence of a holomorphic extension. At first, we will examine the case of a compact set L with empty interior.

Proposition 7.9. *Let L be a compact subset of $\mathbb{C}$ and U be an open subset of $\mathbb{C}$, such that $L \subseteq U$ and $L^\circ = \emptyset$. Let also $f \in H(U \setminus L)$. Then, f is extendable at every $z_0 \in \partial L$ if and only if there exists a holomorphic extension F of f on U. If additionally $f \in A^p(U \setminus L)$ for some $p \in \{0, 1, ...\} \cup \{\infty\}$, then $F \in A^p(U)$.*

Proof. If there exists a holomorphic extension F of f on U, then obviously f is extendable at every $z_0 \in \partial L = L$.

Conversely, if f is extendable at every $z_0 \in L$, then for every $z_0 \in L$ there exist a positive real number r_{z_0} and a holomorphic function F_{z_0} on $D(z_0, r_{z_0})$ such that $D(z_0, r_{z_0}) \subseteq U$ and $F_{z_0}|_{(U \setminus L) \cap D(z_0, r_{z_0})} = f|_{(U \setminus L) \cap D(z_0, r_{z_0})}$. Let $z_1, z_2 \in L$ such that $V = D(z_1, r_{z_1}) \cap D(z_2, r_{z_2}) \neq \emptyset$. Since $L^\circ = \emptyset$, $V \setminus L$ is a non-empty, open set. Thus, F_{z_1}, F_{z_2} are holomorphic on the domain V and coincide with f on $V \setminus L$. By analytic continuation, $F_{z_1} = F_{z_2}$ on V. So, the function F defined on U such that $F(z) = F_z(z)$ for every $z \in L$ and $F(z) = f(z)$ for every $z \in U \setminus L$ is a holomorphic extension of f on U. Obviously, if $f \in A^p(U \setminus L)$, then $F \in A^p(U)$. □

Remark 7.10. If $L^\circ \neq \emptyset$ the equivalence at Proposition 7.9 is not true. Indeed, if $w \in L^\circ \neq \emptyset$, then the holomorphic function $f(z) = \frac{1}{z-w}$ for $z \in U \setminus L$ can not be extended to a holomorphic function on U, but it is extendable at every $z_0 \in \partial L$.

We again consider a compact set $L \subseteq \mathbb{C}$ and an open set $U \subseteq \mathbb{C}$, such that $L \subseteq U$ and a $p \in \{0, 1, 2, \cdots\} \cup \{\infty\}$. Now, we want to find a similar connection between $a_p(L)$ and $a_p(L \cap \overline{D(z_0, r)})$; that is, is the condition $a_p(L) = 0$ equivalent to the condition $a_p(L \cap \overline{D(z_0, r)}) = 0$ for all $z_0 \in L$?

If we suppose that $L^\circ \neq \emptyset$, then there exist z_0 and $r > 0$ such that $D(z_0, r) \subseteq L$. Thus, $a_p(L)$ and $a_p(L \cap \overline{D(z_0, r)})$ are strictly positive.

So, we do not need to assume that $L^\circ = \emptyset$, since it follows from both the conditions $a_p(L) = 0$ and $a_p(L \cap \overline{D(z_0, r)}) = 0$ for every $z_0 \in L$ and for some $r = r_{z_0} > 0$. Also, the first condition obviously implies the second one.

Probably Theorem 3.6 holds even for $p \geq 1$. Specifically, if $a_p(L) = 0$ and V is an open set, then every function $g \in A^p(V \setminus L)$ belongs to $A^p(V)$. This leads us to believe that the above conditions are in fact equivalent. However, this will be examined in future papers.

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References

[1] E. Bolkas, V. Nestoridis and C. Panagiotis *Non extendability from any side of the domain of definition as a generic property of smooth or simply continuous functions on an analytic curve,* arXiv:1511.08584.

[2] F. S. CATER *Differentiable, Nowhere Analytic functions,* Amer. Math. Monthly 91 (1984) no. 10, 618-624.

[3] A. Daghighi and S. Krantz *A note on a conjecture concerning boundary uniqueness,* arxiv 1407.1763v2 [math.CV], 12 Aug. 2015.

[4] T. Gamelin *Uniform Algebras,* AMS Chelsea Publishing, Providence, Rhode Island, 2005.

[5] J. Garnett *Analytic Capacity and Measure,* Lecture Notes in Mathematics, vol. 277, Springer-Verlag Berlin, Heidelberg, New York, 1972.

[6] J. Garnett *Positive length but zero analytic capacity,* Proc. Amer. Math. Soc. 24 (1970), 696-699.

[7] S. Krantz and H. Parks *A primer of Real Analytic Functions,* Second edition, 2002 Birkhauser Boston.

[8] V. Nestoridis *Non extendable holomorphic functions,* Math. Proc. Cambridge Philos. Soc. 139 (2005), no. 2, 351–360.

[9] V. Nestoridis and A. Papadopoulos *Arc length as a conformal parameter for locally analytic curves,* arxiv: 1508.07694.

[10] V. Nestoridis and A. Papadopoulos *Arc length as a global conformal parameter for analytic curves,* JMAA, to appear, D01.10.1016/j, JMAA 2016.02.031.

[11] M. H. A. Newmann, M. A., F. R. S *Elements of the topology of plane sets of points,* Cambridge at the University Press, 1954.

[12] W. Rudin *Function theory in polydiscs,* W. A. Benjamin 1969.

[13] H. Whitney *Analytic extensions of differentiable functions defined in closed sets,* Trans. Amer. Math. Soc. 36 (1934), 63-89.

[14] L. Zalcman *Analytic capacity and rational approximation,* Lecture Notes in Math. No. 50, Springer-Verlag, Berlin, 1968.

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