Math. Z. (2010) 266:169–179
DOI 10.1007/s00209-009-0559-y

Mathematische Zeitschrift

Singular oscillatory integrals on $\mathbb{R}^n$

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Received: 16 January 2009 / Accepted: 12 May 2009 / Published online: 10 June 2009

Abstract Let $\mathcal{P}_{d,n}$ denote the space of all real polynomials of degree at most d on $\mathbb{R}^n$. We prove a new estimate for the logarithmic measure of the sublevel set of a polynomial $P \in \mathcal{P}_{d,1}$. Using this estimate, we prove that

$$\sup_{P\in\mathcal{P}_{d,n}} \left| p.v. \int_{\mathbb{R}^n} e^{iP(x)} \frac{\Omega(x/|x|)}{|x|^n} dx \right| \leq c \log d \, (\|\Omega\|_{L\log L(S^{n-1})} + 1),$$

for some absolute positive constant c and every function Ω with zero mean value on the unit sphere S^{n-1}. This improves a result of Stein (Ann Math Stud 112:307–355, 1986).

Mathematics Subject Classification (2000) Primary 42B20; Secondary 26D05

1 Introduction

We denote by $\mathcal{P}_{d,n}$ the vector space of all real polynomials of degree at most d in $\mathbb{R}^n$. Let K be a $-n$ homogeneous function on $\mathbb{R}^n$, that is,

$$K(x) = \frac{\Omega(x/|x|)}{|x|^n}, \tag{1.1}$$

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where Ω is some function on the unit sphere S^{n-1}. Consider the principal value integral

$$I_n(P) = \left| p.v. \int_{\mathbb{R}^n} e^{iP(x)} K(x) dx \right|.$$

Stein has proved in [4] that if Ω has zero mean value on the unit sphere, then

$$|I_n(P)| \leq c_d \|\Omega\|_{L^\infty(S^{n-1})}, \tag{1.2}$$

for some constant c_d depending on d. We wish to obtain sharp estimates of the form (1.2). The one dimensional analogue, namely the estimate

$$\left| p.v. \int_{\mathbb{R}} e^{iP(x)} \frac{dx}{x} \right| \leq c \log d, \tag{1.3}$$

which was proved in [3], suggests that the constant c_d in (1.2) could be replaced by $c \log d$ for some absolute positive constant c. The fact that this is indeed the case is the content of the following theorem.

Theorem 1.1 *Suppose that $K(x) = \Omega(x/|x|)/|x|^n$ where Ω has zero mean value on the unit sphere S^{n-1}. There exists an absolute positive constant c such that*

$$\sup_{P \in \mathcal{P}_{d,n}} \left| p.v. \int_{\mathbb{R}^n} e^{iP(x)} K(x) dx \right| \leq c \log d \, (\|\Omega\|_{L \log L(S^{n-1})} + 1).$$

Remark 1.2 Suppose that $K(x) = \Omega(x/|x|)/|x|^n$ where the function Ω is odd on the unit sphere. It is an immediate consequence of the one-dimensional result that

$$\sup_{P \in \mathcal{P}_{d,n}} \left| p.v. \int_{\mathbb{R}^n} e^{iP(x)} K(x) dx \right| \leq c \log d \, \|\Omega\|_{L^1(S^{n-1})}$$

for some absolute positive constant c.

The main ingredient of the proof of Theorem 1.1 is an estimate for the logarithmic measure of the sublevel set of a real polynomial in one dimension. This is a lemma of independent interest which we now state.

Lemma 1.3 (The logarithmic measure lemma) *Let $P(x) = \sum_{k=0}^{d} b_k x^k$ be a real valued polynomial of degree at most d, $\alpha > 0$ and $M = \max\{|b_k| : \frac{d}{2} < k \leq d\}$. If $E = \{x \geq 1 : |P(x)| \leq \alpha\}$, then*

$$\int_E \frac{dx}{x} \leq c \min\left(\left(\frac{\alpha}{M}\right)^{\frac{1}{d}}, 1 + \frac{1}{d} \log^+ \frac{\alpha}{M} \right),$$

where c is an absolute positive constant.

Lemma 1.3 should be compared to the following variation of a classical result of Vinogradov which can be found in [6]:

Lemma 1.4 *Let $P(x) = \sum_{k=0}^{d} b_k x^k$ be a real valued polynomial of degree at most d, $\alpha > 0$ and $M_r = \max\{|b_k| : r \leq k \leq d\}$. Let $1 < R$. Then*

$$|\{x \in [1, R] : |P(x)| \leq \alpha\}| \leq cR^{1-\frac{r}{d}} \frac{\alpha^{\frac{1}{d}}}{M_r^{\frac{1}{d}}},$$

where c is an absolute positive constant.

The estimates above depend on the length of the interval $[1, R]$ in all cases but the one where $r = d$. The dependence on R is sharp as can be seen by a scaling argument.

When $r = d$ we get

$$|\{x \in [1, R] : |P(x)| \leq \alpha\}| \leq c \frac{\alpha^{\frac{1}{d}}}{|b_d|^{\frac{1}{d}}}. \tag{1.4}$$

The last inequality corresponds to the following more general result about sublevel sets which was proved in [1]:

Lemma 1.5 *Let ϕ be a C^k function on the interval $[1, R]$ for some $k \geq 1$ and $R > 1$. Suppose that $|\phi^{(k)}(x)| \geq M$ on $[1, R]$. Then*

$$|\{x \in [1, R] : |\phi(x)| \leq \alpha\}| \leq ck \frac{\alpha^{\frac{1}{k}}}{M^{\frac{1}{k}}},$$

where c is an absolute positive constant.

Observe that inequality (1.4) can be deduced by Lemma 1.5 by taking $k = d$ derivatives of the phase function $\phi(x) = P(x)$.

In case $n = 1$ the "linear" part $(\frac{a}{M})^{\frac{1}{d}}$ of the estimate of $\int_E \frac{1}{x} dx$ in Lemma 1.3 is enough for the proof of Theorem 1.1. In fact, the author in [3] used Lemma 1.4 in some appropriate way to prove the above "linear" estimate of Lemma 1.3.

In case $n > 1$ the "logarithmic" part of the estimate of $\int_E \frac{1}{x} dx$ is essential in the proof of Theorem 1.1 as can easily be seen by examining the argument therein.

The structure of the rest of this work is as follows. In Sect. 2 we state some preliminary results. In Sect. 3 we present the proof of Lemma 1.3 and Sect. 3 contains the proof of Theorem 1.1. Finally in Sect. 4 we give a proof of Theorem 1.1 in case $n = 1$ which uses (the "linear" estimate in) Lemma 1.3 and not Lemma 1.4 and which is thus simpler than the proof appearing in [3].

Notation We will use the letter c to denote an absolute positive constant which might change even in the same line of text.

2 Preliminary results

As is usually the case when one deals with oscillatory integrals, a key Lemma is the classical van der Corput Lemma.

Lemma 2.1 (van der Corput) *Let $\phi : [a, b] \to \mathbb{R}$ be a C^1 function and suppose that $|\phi'(t)| \geq 1$ for all $t \in [a, b]$ and ϕ' changes monotonicity N times in $[a, b]$. Then, for every $\lambda \in \mathbb{R}$,*

$$\left|\int_a^b e^{i\lambda\phi(x)}dx\right| \le \frac{cN}{|\lambda|}$$

where c is an absolute constant independent of a,b and ϕ.

The proof of Lemma 2.1 is a simple integration by parts.

We will also need a precise estimate for the Lebesgue measure of the sublevel set of a polynomial on $\mathbb{R}^n$.

Theorem 2.2 (Carbery,Wright) *Suppose that $K \subset \mathbb{R}^n$ is a convex body of volume 1 and $P \in \mathcal{P}_{d,n}$. Let $1 \le q \le \infty$. Then,*

$$|\{x \in K : |P(x)| \le \alpha\}| \le c\min(qd, n)\alpha^{\frac{1}{d}}\|P\|_{L^q(K)}^{-\frac{1}{d}}.$$

This is a consequence of a more general Theorem of Carbery and Wright and can be found in [2].

Corollary 2.3 *Let P be a real homogeneous polynomial of degree k on $\mathbb{R}^n$. Then*

$$\int_{S^{n-1}} \frac{\|P\|_{L^\infty(S^{n-1})}^{\frac{1}{2k}}}{|P(x')|^{\frac{1}{2k}}} d\sigma_{n-1}(x') \le c. \tag{2.1}$$

Proof of Corollary 2.3 Let $B = B(0,\rho)$ be the ball of volume 1 on $\mathbb{R}^n$. For $\epsilon < \frac{1}{k}$ and some $\lambda > 0$ to be defined later, we have

$$\begin{aligned}\int_B |P(x)|^{-\epsilon}dx &= \int_0^\infty |\{x \in B : |P(x)|^{-\epsilon} \ge \alpha\}|d\alpha \\ &\le \lambda + \int_\lambda^\infty |\{x \in B : |P(x)| < \alpha^{-\frac{1}{\epsilon}}\}|d\alpha \\ &\le \lambda + cn\|P\|_{L^\infty(B)}^{-\frac{1}{k}} \frac{\lambda^{-\frac{1}{k\epsilon}+1}}{\frac{1}{k\epsilon}-1},\end{aligned}$$

using Theorem 2.2. Optimizing in λ we get

$$\int_B |P(x)|^{-\epsilon}dx \le \left(cn\frac{k\epsilon}{1-k\epsilon}\right)^{k\epsilon}\|P\|_{L^\infty(B)}^{-\epsilon}.$$

Using polar coordinates and setting $\epsilon = \frac{1}{2k} < \frac{1}{k}$, we then get

$$\begin{aligned}\|P\|_{L^\infty(S^{n-1})}^{\frac{1}{2k}} \int_{S^{n-1}} |P(x')|^{-\frac{1}{2k}} d\sigma_{n-1}(x') &\le c\frac{n^{\frac{3}{2}}}{\rho^n} = c\frac{n^{\frac{3}{2}}\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)} \\ &\le c\frac{n^{\frac{3}{2}}(e\pi)^{\frac{n}{2}}}{\left(\frac{n}{2}+1\right)^{\frac{n+1}{2}}} \le c,\end{aligned}$$

which completes the proof. □

3 The logarithmic measure lemma

The proof of Lemma 1.3 is motivated by an argument of Vinogradov from [6], used to estimate the *Lebesgue* measure of the sublevel set of a polynomial in a bounded interval. We fix a polynomial $P(x) = \sum_{k=0}^{d} b_k x^k$ and look at the set $E = \{x \geq 1 : |P(x)| \leq \alpha\}$. Note that by replacing α with αM in the statement of the lemma, it is enough to consider the case $M = 1$. Since E is a closed set we can find points $x_0, x_1, \ldots, x_d \in E$ such that $x_0 < x_1 < \cdots < x_d$ and

$$\frac{1}{d}\int_E \frac{dx}{x} = \int_{E\cap[x_j,x_{j+1}]} \frac{dx}{x} \leq \log\frac{x_{j+1}}{x_j}, \qquad 0 \leq j \leq d-1.$$

We set $\mu = \int_E \frac{dx}{x}$ and $t = e^{\frac{\mu}{d}} > 1$ and we have that $x_{j+1} \geq t x_j$, $0 \leq j \leq d-1$. The Lagrange interpolation formula is

$$P(x) = \sum_{j=0}^{d} P(x_j)\frac{(x-x_0)\cdots\widehat{(x-x_j)}\cdots(x-x_d)}{(x_j-x_0)\cdots\widehat{(x_j-x_j)}\cdots(x_j-x_d)}, \ x \in \mathbb{R},$$

where $\hat{u}$ means that u is omitted. Thus,

$$b_k = \sum_{j=0}^{d} P(x_j)(-1)^{d-k}\frac{\sigma_{d-k}(x_0,\ldots,\widehat{x_j},\ldots,x_d)}{(x_j-x_0)\cdots\widehat{(x_j-x_j)}\cdots(x_j-x_d)},$$

where σ_l is the l-th elementary symmetric function of its variables. Therefore

$$\begin{aligned}|b_k| &\leq \alpha\sum_{j=0}^{d}\frac{\sigma_{d-k}(x_0,\ldots,\widehat{x_j},\ldots,x_d)}{|x_j-x_0|\cdots\widehat{|x_j-x_j|}\cdots|x_j-x_d|}\\ &= \alpha\sum_{j=0}^{d}\frac{\sigma_k\left(\frac{1}{x_0},\ldots,\widehat{\frac{1}{x_j}},\ldots,\frac{1}{x_d}\right)}{\left(\frac{x_j}{x_0}-1\right)\cdots\left(\frac{x_j}{x_{j-1}}-1\right)\left(1-\frac{x_j}{x_{j+1}}\right)\cdots\left(1-\frac{x_j}{x_d}\right)}\\ &\leq \alpha\sum_{j=0}^{d}\frac{\sigma_k\left(1,\ldots,\widehat{\frac{1}{t}},\ldots,\frac{1}{t^d}\right)}{(t^j-1)\cdots(t-1)(1-\frac{1}{t})\cdots\left(1-\frac{1}{t^{d-j}}\right)}.\end{aligned}$$

It is easy to see that there exists precisely one j, $0 \leq j \leq \frac{d-1}{2} < d$, for which

$$t^{j-1} < \frac{2t^d}{t^{d+1}+1} \leq t^j. \tag{3.1}$$

It is exactly for this j that $(t^j-1)\cdots(t-1)(1-\frac{1}{t})\cdots(1-\frac{1}{t^{d-j}})$ takes its minimum value as j runs from 0 to d. On the other hand we have

$$\sum_{j=0}^{d}\sigma_k\left(1,\ldots,\widehat{\frac{1}{t_j}},\ldots,\frac{1}{t^k}\right) = (d+1-k)\sigma_k\left(1,\ldots,\frac{1}{t^d}\right)$$

and, hence

$$
\begin{aligned}
|b_k| &\le \alpha\,(d+1-k)\sigma_k\left(1,\dots,\frac{1}{t^d}\right)\frac{1}{(t^j-1)\cdots(t-1)\left(1-\frac{1}{t}\right)\cdots\left(1-\frac{1}{t^{d-j}}\right)}\\
&\le \frac{\alpha\,(d+1-k)\binom{d+1}{k}}{1\cdot t\cdots t^k}\frac{1}{(t^j-1)\cdots(t-1)\left(1-\frac{1}{t}\right)\cdots\left(1-\frac{1}{t^{d-j}}\right)}.
\end{aligned}
\tag{3.2}
$$

From (3.1) we easily see that $t^j<2$ and, since $\frac{\log(x-1)}{x}$ is increasing in the interval $(1,2)$, we find

$$
\begin{aligned}
&\log(t-1)+\cdots+\log(t^j-1)\\
&\quad=\frac{t}{t-1}\left(\frac{\log(t-1)}{t}(t-1)+\cdots+\frac{\log(t^j-1)}{t^j}(t^j-t^{j-1})\right)\\
&\quad\ge\frac{t}{t-1}\int_1^{t^j}\frac{\log(x-1)}{x}dx=\frac{t}{t-1}\int_0^{t^j-1}\frac{\log x}{1+x}dx.
\end{aligned}
\tag{3.3}
$$

Similarly, since $\frac{\log(1-x)}{x}$ is decreasing in the interval $(0,1)$ we get

$$
\begin{aligned}
&\log\left(1-\frac{1}{t^{d-j}}\right)+\cdots+\log\left(1-\frac{1}{t}\right)\\
&=\frac{1}{t-1}\left(\frac{\log(1-\frac{1}{t^{d-j}})}{\frac{1}{t^{d-j}}}\left(\frac{1}{t^{d-j-1}}-\frac{1}{t^{d-j}}\right)+\cdots+\frac{\log(1-\frac{1}{t})}{\frac{1}{t}}\left(1-\frac{1}{t}\right)\right)\\
&\ge\frac{1}{t-1}\int_{\frac{1}{t^{d-j}}}^{1}\frac{\log(1-x)}{x}dx=\frac{1}{t-1}\int_0^{1-\frac{1}{t^{d-j}}}\frac{\log x}{1-x}dx.
\end{aligned}
\tag{3.4}
$$

We let

$$
A=\frac{t^d-1}{t^d+1},\quad B=t^j-1,\quad \Gamma=1-\frac{1}{t^{d-j}},
$$

and, obviously, $0<A,B,\Gamma<1$. From (3.3) and (3.4) we have

$$
\begin{aligned}
&\log(t-1)+\cdots+\log(t^j-1)+\log\left(1-\frac{1}{t^{d-j}}\right)+\cdots+\log\left(1-\frac{1}{t}\right)\\
&\ge\frac{t}{t-1}\int_0^{t^j-1}\frac{\log x}{1+x}dx+\frac{1}{t-1}\int_0^{1-\frac{1}{t^{d-j}}}\frac{\log x}{1-x}dx\\
&=\frac{t}{t-1}\int_0^{B}\frac{\log x}{1+x}dx+\frac{1}{t-1}\int_0^{\Gamma}\frac{\log x}{1-x}dx\\
&=-\frac{t}{t-1}B\log\frac{1}{B}-\frac{1}{t-1}\Gamma\log\frac{1}{\Gamma}-O\left(\frac{t}{t-1}B\right)-O\left(\frac{1}{t-1}\Gamma\right).
\end{aligned}
$$

From (3.1) we get $B, \Gamma \leq \frac{t^{d+1}-1}{t^{d+1}+1}$ and, since $\frac{t+1}{t-1}\frac{t^{d+1}-1}{t^{d+1}+1}$ is decreasing in $t \in (1, +\infty)$, we find

$$\frac{t}{t-1}B \leq \frac{t+1}{t-1}\frac{t^{d+1}-1}{t^{d+1}+1} \leq d+1$$

and, similarly,

$$\frac{1}{t-1}\Gamma \leq \frac{t+1}{t-1}\frac{t^{d+1}-1}{t^{d+1}+1} \leq d+1.$$

Therefore

$$\begin{aligned}
&\log(t-1)+\cdots+\log(t^j-1)+\log\left(1-\frac{1}{t^{d-j}}\right)+\cdots+\log\left(1-\frac{1}{t}\right)\\
&\geq -\frac{t}{t-1}B\log\frac{1}{B}-\frac{1}{t-1}\Gamma\log\frac{1}{\Gamma}-cd\\
&\geq -\frac{2}{t-1}A\log\frac{1}{A}-\frac{1}{t-1}\left(B\log\frac{1}{B}+\Gamma\log\frac{1}{\Gamma}-2A\log\frac{1}{A}\right)-cd.
\end{aligned}$$

Now

$$\begin{aligned}
B\log\frac{1}{B}+\Gamma\log\frac{1}{\Gamma}-2A\log\frac{1}{A} &= (B+\Gamma-2A)\log\frac{1}{A}+A\frac{B}{A}\log\frac{A}{B}+A\frac{\Gamma}{A}\log\frac{A}{\Gamma}\\
&\leq \left(\frac{B+\Gamma}{A}-2\right)A\log\frac{1}{A}+cA.
\end{aligned}$$

Using (3.1)

$$\frac{B+\Gamma}{A}-1 \leq \frac{2(t-1)}{t^{d+1}+1}$$

and we conclude that

$$\begin{aligned}
\frac{1}{t-1}\left(B\log\frac{1}{B}+\Gamma\log\frac{1}{\Gamma}-2A\log\frac{1}{A}\right) &\leq \frac{2}{t^{d+1}+1}A\log\frac{1}{A}+\frac{c}{t-1}A\\
&\leq c+c\frac{t+1}{t-1}\frac{t^d-1}{t^d+1} \leq cd.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\log(t-1)+\cdots+\log(t^j-1)+\log\left(1-\frac{1}{t^{d-j}}\right)+\cdots+\log\left(1-\frac{1}{t}\right)\\
&\geq -\frac{2}{t-1}A\log\frac{1}{A}-cd
\end{aligned}$$

and, finally, (3.2) implies that for some $k > \frac{d}{2}$

$$1 \leq \frac{c_o^d\alpha}{t^{\frac{k(k-1)}{2}}}\left(\frac{1}{A}\right)^{\frac{2A}{t-1}},$$

where c_o is an absolute positive constant.

Case 1 $c_o\alpha^{\frac{1}{d}} < \frac{1}{2}$. Then, since $\frac{2A}{t-1} \leq \frac{t+1}{t-1}A \leq d$, we get

$$A^d \leq A^{\frac{2A}{t-1}} \leq c_o^d\alpha$$

which implies

$$\frac{t^d - 1}{t^d + 1} = A \leq c_o\, \alpha^{\frac{1}{d}}$$

and, finally,

$$\mu \leq e^{\mu} - 1 = t^d - 1 \leq 4c_o\alpha^{\frac{1}{d}}.$$

Case 2 $c_o\alpha^{\frac{1}{d}} \geq \frac{1}{2}$, $t^d < 2$. Then

$$1 < e^{\mu} = t^d < 4c_o\alpha^{\frac{1}{d}}$$

which shows that

$$\mu < \log^+(4c_o) + \frac{\log^+\alpha}{d}.$$

Case 3 $c_o\alpha^{\frac{1}{d}} \geq \frac{1}{2}$, $t^d \geq 2$. Then $A \geq \frac{1}{3}$ and $\frac{2A}{t-1} \leq \frac{t+1}{t-1}A \leq d$ and, hence,

$$\frac{1}{3^d} t^{\frac{k(k-1)}{2}} \leq c_o^d \alpha.$$

We conclude that

$$\mu \leq \frac{2d^2}{k(k-1)}\left(\log^+(3c_o) + \frac{\log^+\alpha}{d}\right) \leq c\left(1 + \frac{\log^+\alpha}{d}\right)$$

since $k > \frac{d}{2}$.

Proof of Theorem 1.1 Let Ω be a function with zero mean value on the unit sphere S^{n-1} belonging to the class $L\log L(S^{n-1})$, that is

$$\|\Omega\|_{L\log L(S^{n-1})} = \int_{S^{n-1}} |\Omega(x')|(1 + \log^+|\Omega(x')|)d\sigma_{n-1}(x') < \infty.$$

Set $K(x) = \Omega(x/|x|)/|x|^n$ and let $P \in \mathcal{P}_{d,n}$. We will show the theorem for $d = 2^m$, for some $m \geq 0$. The general case is then an immediate consequence.

We set

$$C_d = \sup_{\substack{0<\epsilon<R \\ P\in\mathcal{P}_{d,n}}} \left| \int_{\epsilon\leq|x|\leq R} e^{iP(x)}K(x)dx \right|,$$

where C_d is a constant depending on d, Ω and n. For $0 < \epsilon < R$ and $P \in \mathcal{P}_{d,n}$ we write,

$$I_{\epsilon,R}(P) = \int_{\epsilon\leq|x|\leq R} e^{iP(x)}K(x)dx = \int_{S^{n-1}} \int_{\epsilon}^{R} e^{iP(rx')}\frac{dr}{r}\Omega(x')d\sigma_{n-1}(x').$$

For $x' \in S^{n-1}$, we have that $P(rx') = \sum_{j=1}^{d} P_j(x')r^j$ where P_j is a homogeneous polynomial of degree j. Observe that we can omit the constant term, without loss of generality. Set also $m_j = \|P_j\|_{L^\infty(S^{n-1})}$. Since ϵ and R are arbitrary positive numbers, by a dilation in r we can assume that $\max_{\frac{d}{2}<j\le d} m_j = 1$ and, in particular, that $m_{j_o} = 1$ for some $\frac{d}{2} < j_o \le d$. We also write $Q(x) = \sum_{j=1}^{\frac{d}{2}} P_j(x)$. We split the integral in two parts as follows

$$|I_{\epsilon,R}(P)| \le \left| \int\limits_{S^{n-1}} \int\limits_{\epsilon}^{1} e^{iP(rx')} \frac{dr}{r} \Omega(x') d\sigma_{n-1}(x') \right| + \left| \int\limits_{S^{n-1}} \int\limits_{1}^{R} e^{iP(rx')} \frac{dr}{r} \Omega(x') d\sigma_{n-1}(x') \right| = I_1 + I_2.$$

For I_1 we have that

$$\begin{aligned} I_1 &\le \int\limits_{S^{n-1}} \int\limits_{0}^{1} \left| e^{iP(rx')} - e^{iQ(rx')} \right| \frac{dr}{r} |\Omega(x')| d\sigma_{n-1}(x') \\ &\quad + \left| \int\limits_{S^{n-1}} \int\limits_{\epsilon}^{1} e^{iQ(rx')} \frac{dr}{r} \Omega(x') d\sigma_{n-1}(x') \right| \\ &\le \sum_{\frac{d}{2}<j\le d} \frac{m_j}{j} \|\Omega\|_{L^1(S^{n-1})} + C_{\frac{d}{2}} \le c\|\Omega\|_{L^1(S^{n-1})} + C_{\frac{d}{2}}. \end{aligned}$$

For I_2 we write

$$\begin{aligned} I_2 &\le \int\limits_{S^{n-1}} \left| \int\limits_{\{r\in[1,R]:|\frac{\partial P(rx')}{\partial r}|>d\}} e^{iP(rx')} \frac{dr}{r} \right| |\Omega(x')| d\sigma_{n-1}(x') \\ &\quad + \int\limits_{S^{n-1}} \int\limits_{\{r\in[1,R]:|\frac{\partial P(rx')}{\partial r}|\le d\}} \frac{dr}{r} |\Omega(x')| d\sigma_{n-1}(x'). \end{aligned}$$

Since $\{r \in [1, R] : |\frac{\partial P(rx')}{\partial r}| > d\}$ consists of at most $O(d)$ intervals where $\frac{\partial P(rx')}{\partial r}$ is monotonic, a simple corollary to van der Corput's lemma for the first derivative [5, Corollary on p. 334] gives the bound

$$\int\limits_{S^{n-1}} \left| \int\limits_{\{r\in[1,R]:|\frac{\partial P(rx')}{\partial r}|>d\}} e^{iP(rx')} \frac{dr}{r} \right| |\Omega(x')| d\sigma_{n-1}(x') \le c\|\Omega\|_{L^1(S^{n-1})}.$$

On the other hand, the logarithmic measure lemma implies that

$$\begin{aligned} &\int\limits_{S^{n-1}} \int\limits_{\{r\in[1,R]:|\frac{\partial P(rx')}{\partial r}|\le d\}} \frac{dr}{r} |\Omega(x')| d\sigma_{n-1}(x') \\ &\le c\|\Omega\|_{L^1(S^{n-1})} + c\frac{1}{d} \int\limits_{S^{n-1}} \log \frac{d}{\max_{\frac{d}{2}<j\le d}\{j|P_j(x')|\}} |\Omega(x')| d\sigma_{n-1}(x'). \end{aligned}$$

Combining the estimates we get

$$C_d \le c\|\Omega\|_{L^1(S^{n-1})} + C_{\frac{d}{2}} + c\frac{2j_o}{d}\int\limits_{S^{n-1}} \log \frac{\|P_{j_o}\|^{\frac{1}{2j_o}}_{L^\infty(S^{n-1})}}{|P_{j_o}(x')|^{\frac{1}{2j_o}}}|\Omega(x')|d\sigma_{n-1}(x')$$

and, from Young's inequality,

$$C_d \le c\|\Omega\|_{L^1(S^{n-1})} + C_{\frac{d}{2}} + c\int\limits_{S^{n-1}} \frac{\|P_{j_o}\|^{\frac{1}{2j_o}}_{L^\infty(S^{n-1})}}{|P_{j_o}(x')|^{\frac{1}{2j_o}}}d\sigma_{n-1}(x')$$
$$+ c\int\limits_{S^{n-1}} |\Omega(x')|(1+\log^+|\Omega(x')|)d\sigma_{n-1}(x').$$

Now, using Corollary 2.3 we get

$$C_d \le C_{\frac{d}{2}} + c(\|\Omega\|_{L\log L(S^{n-1})} + 1).$$

Since $d = 2^m$, this means that

$$C_{2^m} \le C_{2^{m-1}} + c(\|\Omega\|_{L\log L(S^{n-1})} + 1).$$

Using induction on m we get that $C_{2^m} \le C_1 + cm(\|\Omega\|_{L\log L(S^{n-1})} + 1)$. Observe that C_1 corresponds to some polynomial $P(x) = b_1x_1 + \cdots + b_nx_n$. We write

$$\left|\int\limits_{\epsilon<|x|<R} e^{iP(x)}K(x)dx\right|$$
$$= \left|\int\limits_{S^{n-1}}\int\limits_{\epsilon}^{R}\{e^{irP(x')} - e^{ir\|P\|_{L^\infty(S^{n-1})}}\}\frac{dr}{r}\Omega(x')d\sigma_{n-1}(x')\right|.$$

Using the simple estimate

$$\left|\int\limits_{\epsilon}^{R}\{e^{iar} - e^{ibr}\}\frac{dr}{r}\right| \le c + c\left|\log\left|\frac{b}{a}\right|\right|$$

we get

$$\left|\int\limits_{\epsilon<|x|<R} e^{iP(x)}K(x)dx\right| \le c\|\Omega\|_{L^1(S^{n-1})} + c\int\limits_{S^{n-1}} \log\frac{\|P\|^{\frac{1}{2}}_{L^\infty(S^{n-1})}}{|P(x')|^{\frac{1}{2}}}|\Omega(x')|d\sigma_{n-1}(x').$$

Hence, $C_1 \le c\|\Omega\|_{L^1(S^{n-1})} + c + \|\Omega\|_{L\log L(S^{n-1})}$ and

$$C_{2^m} \le cm(\|\Omega\|_{L\log L(S^{n-1})} + 1).$$

The case of general d is now trivial. If $2^{m-1} < d \le 2^m$ then

$$C_d \le C_{2^m} \le cm(\|\Omega\|_{L\log L(S^{n-1})} + 1) \le c\log d(\|\Omega\|_{L\log L(S^{n-1})} + 1).$$

4 The one dimensional case revisited

We will attempt to give a short proof of the one dimensional analogue of Theorem 1.1. This is a slight simplification of the proof in [3], with the aid of the logarithmic measure lemma.

So, fix a real polynomial $P(x) = b_0 + b_1 x + \cdots + b_d x^d$ and consider the quantity

$$C_d = \sup_{0<\epsilon<R} \left| \int_{\epsilon<|x|<R} e^{iP(x)} \frac{dx}{x} \right|.$$

By the same considerations as in the n-dimensional case, we can assume that P has no constant term and that it can be decomposed in the form

$$P(x) = \sum_{0<j\le\frac{d}{2}} b_j x^j + \sum_{\frac{d}{2}<j\le d} b_j x^j = Q(x) + R(x),$$

where $\max_{\frac{d}{2}<j\le d} |b_j| = 1$. As a result

$$\left| \int_{\epsilon<|x|<R} e^{iP(x)} \frac{dx}{x} \right| \le C_{\frac{d}{2}} + \int_{0<|x|<1} \frac{|R(x)|}{x} dx + \left| \int_{1<|x|<R} e^{iP(x)} \frac{dx}{x} \right|$$
$$\le C_{\frac{d}{2}} + c + I.$$

We split I as follows

$$I \le \left| \int_{\{x\in[1,R):|P'(x)|>d\}} e^{iP(x)} \frac{dx}{x} \right| + \int_{\{x\ge1:|P'(x)|\le d\}} \frac{dx}{x}.$$

Now, using Proposition 2.1 for the first summand in the above estimate and the logarithmic measure lemma to estimate the second summand, we get that $I \le c$. But this means that $C_d \le C_{\frac{d}{2}} + c$ which completes the proof by considering first the case $d = 2^m$ for some m, as in the n-dimensional case.

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