Concentration of mass on isotropic convex bodies

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Abstract

We establish sharp concentration of mass for isotropic convex bodies: there exists an absolute constant $c > 0$ such that if K is an isotropic convex body in $\mathbb{R}^n$, then

$$\mathrm{Prob}\left(\left\{x \in K : \|x\|_2 \geq c\sqrt{n}L_K t\right\}\right) \leq \exp\left(-\sqrt{n}t\right)$$

for every $t \geq 1$, where L_K denotes the isotropic constant.

1. Introduction

A convex body K in $\mathbb{R}^n$, with volume equal to 1 and center of mass at the origin, is called isotropic if its inertia matrix is a multiple of the identity. Equivalently, if there exists a positive constant L_K such that $\int_K \langle x, \theta\rangle^2 dx = L_K^2$ for every $\theta \in S^{n-1}$. The starting point of this paper is the following concentration estimate of Alesker [1]: If K is an isotropic convex body in $\mathbb{R}^n$ then, for every $t \geq 1$ we have

$$\mathrm{Prob}\left(\left\{x \in K : \|x\|_2 \geq c\sqrt{n}L_K t\right\}\right) \leq 2\exp(-t^2).$$

Throughout this note, we write B_2^n for the Euclidean unit ball and $\|\cdot\|_2$ for the Euclidean norm; c, c_1, c_2 etc. will denote absolute positive constants.

Bobkov and Nazarov (see [2] and [3]) have obtained a striking strengthening of Alesker's estimate for the class of 1-unconditional isotropic convex bodies: in this case,

$$\mathrm{Prob}\left(\left\{x \in K : \|x\|_2 \geq c\sqrt{n}L_K t\right\}\right) \leq \exp\left(-\sqrt{n}t\right)$$

for every $t \geq 1$. Strong dimension dependent volume concentration was recently confirmed in [5] for the unit balls of the Schatten trace classes as well.

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The purpose of this note is to establish the fact that the "Bobkov-Nazarov estimate" holds true in full generality.

Theorem 1.1 *If K is an isotropic convex body in $\mathbb{R}^n$ then, for every $t \geq 1$ we have that*

$$\mathrm{Prob}\left(\left\{x \in K : \|x\|_2 \geq c\sqrt{n} L_K t\right\}\right) \leq \exp\left(-\sqrt{n} t\right).$$

2. Sketch of the Proof

Let K be an isotropic convex body in $\mathbb{R}^n$. For any $q \geq 1$, we define $I_q(K) := \left(\int_K \|x\|_2^q dx\right)^{1/q}$. As observed in [11], in order to prove Theorem 1.1 it is enough to show that

$$I_{c_1\sqrt{n}}(K) \leq c_2 I_2(K). \tag{1}$$

For every $q \geq 1$ we define the L_q-centroid body $Z_q(K)$ of K by its support function $h_{Z_q(K)}(y) := \left(\int_K |\langle x, y\rangle|^q dx\right)^{1/q}$. Under a different normalization these bodies were introduced in [7]. The family $\{Z_q(K) : q \geq 1\}$ increases to the body $Z_\infty(K) = \mathrm{conv}\{K, -K\}$. Since K is isotropic, we have $Z_2(K) = L_K B_2^n$. We will use the following facts:

(i) Let C be a symmetric convex body in $\mathbb{R}^n$ and let $w_q(C) := \left(\int_{S^{n-1}} h_C^q(\phi) d\sigma(\phi)\right)^{1/q}$ be the q-th mean width of C. It is easily checked that

$$w_q(Z_q(K)) \simeq \sqrt{\frac{q}{q+n}} I_q(K). \tag{2}$$

(ii) In [6] it is proved that if $k_*(C)$ is the largest positive integer for which $\mu_{n,k}\{F \in G_{n,k} : \frac{1}{2} w_1(C)\|x\|_2 \leq h_C(x) \leq 2w_1(C)\|x\|_2$ for all $x \in F\} \geq 1 - \frac{k}{n-k}$, then

$$w_1(C) \simeq w_q(C) \tag{3}$$

for every $q \leq k_*(C)$. Here $G_{n,k}$ is the Grassmann manifold of k-dimensional subspaces of $\mathbb{R}^n$ equipped with the Haar probability measure $\mu_{n,k}$. Recall that the critical dimension k_* is completely determined by the mean width $w_1(C)$ and the circumradius $R(C)$ of C; in [10] it is shown that $k_*(C) \simeq n\frac{W(C)^2}{R(C)^2}$.

(iii) Definition. Let K be a convex body of volume 1 in $\mathbb{R}^n$. We define

$$q_* := q_*(K) := \max\{q \in \mathbb{N} : k_*(Z_q^\circ(K)) \geq q\},$$

where $Z_q^\circ(K)$ is the polar body of $Z_q(K)$. A related parameter was introduced in [12], where the following lower bound for $q_*(K)$ was also proved.

Proposition 2.1 *If K is an isotropic convex body in $\mathbb{R}^n$ then*

$$q_*(K) \geq c\sqrt{n}. \tag{4}$$

From the above discussion it becomes clear that (1), and hence Theorem 1.1, will follow if we show that

$$w_{q_*}(Z_{q_*}(K)) \leq c\sqrt{q_*} L_K \quad \text{or, equivalently,} \quad w(Z_{q_*}(K)) \leq c\sqrt{q_*} L_K. \tag{5}$$

By the definition of $q_*(K)$ there exists $F \in G_{n,q_*}$ such that $\frac{1}{2} w(Z_{q_*}(K)) \leq h_{Z_{q_*}(K)}(\theta) \leq 2w(Z_{q_*}(K))$ for all $\theta \in S_F := S^{n-1} \cap F$. The following proposition completes the proof:

Proposition 2.2 *Let K be an isotropic convex body in $\mathbb{R}^n$. For every integer $q \geq 1$ and every $F \in G_{n,q}$ there exists $\theta \in S_F$ such that*

$$h_{Z_q(K)}(\theta) \leq c\sqrt{q}\, L_K. \tag{6}$$

Sketch of the proof. Fix $F \in G_{n,q}$ of and write E for the orthogonal subspace of F and P_F for the orthogonal projection onto F. For every $\phi \in S_F$ we define $E^+(\phi) = \{x \in \text{span}\{E, \phi\} : \langle x, \phi\rangle \geq 0\}$.

Let $q \geq 0$ and write $B_q(K, F)$ for the convex body in F defined by the gauge function

$$\phi \mapsto \|\phi\|_2^{1+\frac{q}{q+1}} \left(\int_{K \cap E^+(\phi)} |\langle x, \phi\rangle|^q dx \right)^{-1/(q+1)}$$

(see [9] for details and references). Integration in polar and cylindrical coordinates shows that

$$P_F(Z_q(K)) = q^{1/q} |B_{2q-1}(K, F)|^{2/q} Z_q(\overline{B}_{2q-1}(K, F)), \tag{7}$$

where $\overline{A}$ denotes the homothet $A/|A|^{1/n}$ of volume 1 of a convex body A. Using well known Khintchine type inequalities for log-concave functions (see [9] for details and references) we get

$$|B_{2q-1}(K, F)|^{2/q} \leq c \frac{L_K}{L_{\overline{B}_{q+1}(K,F)}}. \tag{8}$$

Therefore,

$$P_F(Z_q(K)) \subseteq c_1 \frac{L_K}{L_{\overline{B}_{q+1}(K,F)}} Z_q(\overline{B}_{2q-1}(K, F)) \subseteq c_2 L_K Z_\infty(\overline{B}_{2q-1}(K, F)). \tag{9}$$

Taking volumes in (9) and estimating the volume of $Z_\infty(\overline{B}_{2q-1}(K, F))$ by a standard use of the Rogers-Shephard inequality, we complete the proof.

It is interesting to note that the estimate of Theorem 1.1 is sharp in both n and t; the ℓ_1^n-ball B_1^n is the extremal isotropic convex body in the following sense: For every isotropic convex body K in $\mathbb{R}^n$ and for every $2 \leq q \leq \infty$,

$$\frac{I_q(K)}{I_2(K)} \leq c \frac{I_q(\overline{B_1^n})}{I_2(\overline{B_1^n})}.$$

3. Further Results

(i) *Reverse L_q-affine isoperimetric inequality.* Lutwak, Yang and Zhang proved in [8] that if K is a convex body of volume 1 in $\mathbb{R}^n$, then

$$|Z_q(K)|^{1/n} \geq |Z_q(\overline{B_2^n})|^{1/n} \geq c\sqrt{q/n}$$

for every $1 \leq q \leq n$, where $c > 0$ is an absolute constant. Our analysis of the L_q-centroid bodies leads to the following reverse inequality.

Theorem 3.1 *Let K be a convex body of volume 1 in $\mathbb{R}^n$. For every $1 \leq q \leq n$ we have that*

$$|Z_q(K)|^{1/n} \leq c\sqrt{q/n}\, L_K,$$

where $c > 0$ is a universal constant.

(ii) *Random points in isotropic convex bodies.* Let $\varepsilon \in (0, 1)$ and consider N independent random points $x_1, \ldots, x_N$ uniformly distributed in an isotropic convex body K in $\mathbb{R}^n$. A question of Kannan, Lovász and Simonovits is to find N_0, as small as possible, for which the following holds true: if $N \geq N_0$ then with probability greater than $1 - \varepsilon$ one has $\left\| \text{Id} - \frac{1}{NL_K^2} \sum_{i=1}^N x_i \otimes x_i \right\| \leq \varepsilon$. Bourgain in [4] proved that one can choose $N_0 \simeq c(\varepsilon) n (\log n)^3$; this was improved to $N_0 \simeq c(\varepsilon) n (\log n)^2$ by Rudelson [13]. Theorem 1.1 allows us to remove one more logarithmic term.

Theorem 3.2 *Let $\varepsilon \in (0,1)$ and let K be an isotropic convex body in $\mathbb{R}^n$. If $N \geq c(\varepsilon) n \log n$, and if $x_1, \ldots, x_N$ are independent random points uniformly distributed in K, then with probability greater than $1-\varepsilon$ we have*

$$(1-\varepsilon)L_K^2 \leq \frac{1}{N}\sum_{i=1}^{N}\langle x_i, \theta\rangle^2 \leq (1+\varepsilon)L_K^2$$

for every $\theta \in S^{n-1}$.

Concluding Remark. All the results of this note remain valid if we replace Lebesgue measure on an isotropic convex body by an arbitrary isotropic log-concave measure. Detailed references, proofs and various extensions of the results of this note will appear elsewhere.

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