Calibrations for a free-discontinuity problem with Robin condition

Camille Labourie

Abstract

Free-discontinuity problems consist in minimizing the energy of a pair (u, K) composed of a function $u \in C^1(\mathbf{R}^n \setminus K)$ and a $(n-1)$-dimensional set K. The energy involves a competiton between a Dirichlet-type energy of u in $\mathbf{R}^n \setminus K$ and a surface energy of K. We interpret K as an hypersurface (with possible singularities) where the function u jumps between different values.

We present a free-discontinuity problem inspired by the energy of a thermal insulation configuration. This problem has been studied by Caffarelli–Kriventsov and Bucur–Giacomini by relaxing the functional in the SBV space. It consists then in minimizing

$$E(u) = \int |\nabla u|^2 \, \mathrm{d}\mathcal{L}^n + \int_{J_u} (u^-)^2 + (u^+)^2 \, \mathrm{d}\mathcal{H}^{n-1} + \mathcal{L}^n(\{\, u > 0 \,\})$$

among functions $u \in SBV(\mathbf{R}^n)$ such that $u = 1$ on a fixed domain $\Omega \subset \mathbf{R}^n$.

In contrary to convex functionals, a competitor u which satisfy Euler-Lagrange equations may not be a global minimizer (neither a local minimizer). In [3], Alberti, Bouchitte and Dal Maso have proposed convex relaxations for free-discontinuity functionals. The fact that u minimizes the relaxation is (roughly) characterized by the existence of a specific vector field called *calibration*. In practice, we don't know if a minimizer of $E(u)$ always admit a calibration and such vector-field may be very difficult to build.

The goal of the colloquium is to show how to find sufficient conditions of minimality for our problem by the calibration method. We will first explain the SBV space and introduce the energy $E(u)$. Then we will present known results about the minimizers. Finally, we will define the calibrations for $E(u)$ and we will show examples and applications.

References

[1] L. A. Caffarelli and D. Kriventsov A free boundary problem related to thermal insulation. Comm. Partial Differential Equations 41 (2016), no. 7, 1149-1182.

[2] D. Bucur and A. Giacomini Shape optimization problems with Robin conditions on the free boundary. Ann. Inst. H. Poincaré Anal. Non Linéaire 33 (2016), no. 6, 1539–1568.

[3] G. Alberti; G. Bouchitte; G. Dal Maso The calibration method for the Mumford-Shah functional and free-discontinuity problems. Calc. Var. Partial Differential Equations 16 (2003), no. 3, 299-333.

[4] C. Labourie, E. Milakis, The calibration method for the Thermal Insulation functional Submitted. Available https://arxiv.org/abs/2106.04955